A TEXT BOOK
OF
DIFFERENTIAL EQUATIONS
for
Post-graduate Students

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PREFACE TO THE THIRD EDITION

The call for a new edition indicates that the book has proved itself popular and useful. We are very thankful to the students and the teachers of the subject.

Minor changes have been made. Many new questions have been selected and put at proper places.

Agra, August 1962. AUTHORS.

PREFACE TO THE SECOND EDITION

The book has been thoroughly revised with additions here and there. More examples from University Papers are worked out.

Agra, July 1960. AUTHORS.

PREFACE TO THE FIRST EDITION

This book has been written for the use of students preparing for M. A. and M. Sc. examinations of the Indian Universities and is intended to cover the courses commonly prescribed for these examinations. Great care has been taken to explain the fundamental principles fully and rigorously. To illustrate these principles many examples have been worked out and numerous examples for solution have been added after each important article. These examples have usually been selected from the question papers of various universities and from standard books on the subject. The book is complete as far as it goes and it is hoped nothing of importance has been omitted. Authors will consider their labour highly rewarded if this book is of some value to those for whom it is intended.

We shall be failing in duty if we do not express our thanks to Prof. C. R. Chaturvedi, Ex-Dean Faculty of Science Agra University, who co-ordinated the work, filling in gaps where necessary and making valuable suggestions.

Suggestions for improvement will be thankfully received.

Agra, Sep. 21, 1958. AUTHORS.

CONTENTS

CHAPTER I

INTRODUCTION AND DEFINITIONS

1·1. Origin. When we discuss the theory of bending of beams, oscillations of mechanical systems and of electric currents, conduction of heat, velocity of chemical reactions, diffusions of solvents etc., we come across equations which contain, besides the dependent and independent variables, different derivatives of the dependent variable with respect to the independent variable or variables. Many problems in Mechanics and some other branches of Mathematics can be easily handled with the help of such equations. These equations are called **Differential Equations** and are of the type

$$\frac{d^2y}{dt^2}=k^2y, \qquad \ldots\ldots(1)$$

$$\frac{\partial^3 y}{\partial t^3}=a^2\ \frac{\partial^2 y}{\partial x^2} \qquad \ldots\ldots(2)$$

$$\frac{d^2y}{dx^2}=k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}, \qquad \ldots\ldots(3)$$

$$\frac{d^2y}{dx^2}=\frac{x^{1/3}}{y\ (1+x^{1/2})} \qquad \ldots\ldots(4)$$

$$\frac{dy}{dx}+Py=f\ (x). \qquad \ldots\ldots(5)$$

The differential equations given in (1), (3), (4) and (5) involve only one independent variable and are called **Ordinary Differential Equations.**

The equation given in (2) involves partial derivatives with respect to two independent variables t and x. The equations which contain two or more independent variables and *partial* derivatives with respect to them are called **Partial Differential Equations.**

If a differential equation contains n^{th} and lower derivatives, it is said to be of n^{th} **Order.** Thus equations given in (1), (3) and (4) are of second order while that given in (5) is of first order.

The degree of an equation is the greatest exponent of the highest order derivative when the equation has been *made rational*

and integral as far as the derivatives are concerned. Thus equation (3) must be squared to rationalise it and then we find that the greatest degree of $\frac{d^2y}{dx^2}$, the highest order derivative, is two Hence the equation is of second degree. The equations in (1) and (5) are ordinary differential equations of first degree and of second and first orders respectively.

Note that the definition of degree does not require x or y to be made rational or integral.

Any relation between the dependent and independent variables which, when substituted in the differential equation, reduces it to an identity is called a **Solution** of the differential equation. A solution of a differential equation does not contain the derivatives of the dependent variable with respect to the independent variable or variables.

1·2. Formation of differential Equations. We have already indicated that differential equations have been obtained as a bye-product of the study of certain problems in Physics, Chemistry and some branches of Mathematics. Here we show that they are also obtained as a result of elimination of constants.* This latter method of obtaining the differential equations gives us an idea as to what type of solution a differential equation will have. Consider

$$y=Ax \qquad \text{......(6)}$$

where A is an arbitrary constant.

Differentiating (6) we get

$$\frac{dy}{dx}=A.$$

Eliminating the arbitrary constant A with the help of (6), we get

$$\frac{dy}{dx}=\frac{y}{x}, \qquad \text{......(7)}$$

which is an ordinary differential equation of first order and first degree. If we substitute the value of y from (6) in (7), we get an identity $A=A$, and hence (6) is a solution of (7). Thus we see that the solution of an ordinary differential equation of first order and first degree will consist of one arbitrary constant.

Next, let us take the relation

$$y=A\cos(x+\alpha) \qquad \text{......(8)}$$

*We have taken the case of ordinary differential equations only. Partial differential equations can be obtained by the elimination of arbitrary constants or of arbitrary functions.

Differentiating this we get

$$\frac{dy}{dx}=-A \sin (x+\alpha),$$

$$\frac{d^2y}{dx^2}=-A \cos (x+\alpha).$$

Eliminating the arbitrary constants A and α with the help of (8), we get

$$\frac{d^2y}{dx^2}=-y, \qquad \dots(9)$$

which is a second order equation. Hence, as before, we can say that the solution of a second order equation will consist of two arbitrary constants.

In general, if we have an equation

$$f(x, y, a, b, c, \dots\dots\dots k)=0,$$

containing n arbitrary constants, then differentiating it once, twice,......n times we get n equations. Between these n equations and the given equation, we can eliminate the n constants a, b, c......k to give us a differential equation of n^{th} order.*

This indicates that a solution of the n^{th} order differential equation should contain n arbitrary constants.

Manipulation of Arbitrary Constants. An equation involving arbitrary constants may be written in various forms. Thus $\log \sin x-\log (1-y)=c$ can also be written as $y=1+c_1 \sin x$. Again $\sin^{-1} x+\sin^{-1} y=c$, can be written as $x\sqrt{1-y^2}+y\sqrt{1-x^2}=c_1$, by putting $c=\sin^{-1} c_1$ and taking sine of both sides of the equation.

This shows that the solution of a differential equation may sometimes have different forms.

1·3. Definitions. The solution of an ordinary differential equation of n^{th} order which contains n arbitrary constants is called the **Complete Primitive** or the **General Solution.**

Any solution which is obtained from the complete Primitive by giving *particular* values to the arbitrary constants is called a **Particular Integral.**

The Complete Primitive of (9) is given by (8) which can also be written as

$$y=A \cos x. \cos \alpha-A \sin \alpha. \sin x$$

or

$$y=a \cos x+b \sin x, \qquad \dots(10)$$

*In exceptional cases a relation containing n arbitrary constants may give rise to a differential equation of order less than n.

indicating that the Complete Primitive can be obtained in different forms but the number of constants must be equal to the order of the differential equation. Now $y=\cos x$, is obtained from the Complete Primitive by giving to the arbitrary constants a and b the particular values 1 and 0 respectively. Hence this is a **Particular Integral.** Another Particular Integral is

$$y=\sin x.$$

Note. *In some exceptional cases we find a solution of a differential equation which cannot be derived from the Complete Primitive in this way. Because of this singularity such a solution is called a* **Singular Solution** *and will be dealt with in a separate chapter.*

In many cases the Complete Primitive consists of the terms containing the arbitrary constants and a definite function of the independent variable For facility of solution the Complete Primitive is divided into two parts. The terms which contain the arbitrary constants are called the **Complementary Function** and the remaining part which can be obtained by giving the value zero to each of the arbitrary constants is called the **Particular Integral.** Thus if the Complete Primitive is given by

$$y=e^x+A \sin 2x+B \cos 2x,$$

then e^x is the Particular Integral and $A \sin 2x+B \cos 2x$ is the Complementary Function.

Thus we can say that

"The Complete Primitive or the general solution of a linear differential equation is the sum of the Complementary Function and the Particular integral."

This will be treated in the third chapter.

1·4. Differentials. In the application of the calculus in Geometry, Physics etc., it is generally more convenient to write equations not in terms of the *increments* δx, δy of the functions x and y but in terms of dx, dy, which are called the **differentials** of x and y.

If y a function of only one variable x *i.e.* $y=f(x)$, and further if $f(x)$ is differentiable, then we know that

$$\delta y=\{f'(x)+\varepsilon\}\,\delta x, \qquad \ldots\ldots(1)$$

where $\varepsilon \to 0$ as $\delta x \to 0$. For small values of δx, the increment in y *i. e.* δy is only *approximately* equal to $f'(x)\,\delta x$. We define the isolated symbol dy by the relation

$$dy=f'(x)\,\delta x. \qquad \ldots\ldots(2)$$

It should be noted that the *increment* in y is δy and is given by relation (1) while the **differential** of y is dy and is given by relation (2).

When y is equal to x, we have from (2)

$$dy=dx=\delta x, \text{ as } f'(x)=1. \qquad \ldots\ldots(3)$$

Hence we get from (2) and (3)

$$dy=f'(x)\,dx \qquad \ldots\ldots(4)$$

Thus, we have $\dfrac{dy}{dx}=f'(x)$, $\qquad \ldots\ldots(5)$

which shows that $f'(x)$ is the quotient of the differentials dy, dx.

Thus we find that $\dfrac{dy}{dx}$ acquires a double meaning. In the first place it means $\dfrac{d}{dx}(y)$ where $\dfrac{d}{dx}$ is a symbol; in the second it means the quotient of dy, dx.

However, there is no inconvenience in this since the relation (5) is true whichever meaning we choose.

EXERCISE I

1. Determine the differential equations whose general solutions are given below. [c_1, c_2 and c are arbitrary constants].

   (a) $y=c_1 \sin mx+c_2 \cos mx$ (b) $y=c_1 x+c_2/x$
   (c) $y=cx+c-c^3$ (d) $y^3-2cx-c^2=0$.

2. Show that $4y=c_1x^5+c_2x+1/3x$ is a solution of

$$x^2 \frac{d^2y}{dx^2}-5x\frac{dy}{dx}+5y=1/x.$$

3. Show that $y=c_1+c_2/r$ is a solution of

$$\frac{d^2y}{dr^2}+\frac{2}{r}\frac{dy}{dr}=0.$$

4. Eliminate a and b from the relations

   (a) $y=a \log x+b$ (b) $ay=ae^x+be^{-x}$.

CHAPTER II

EQUATIONS OF FIRST ORDER AND FIRST DEGREE

2·1. The differential equations of first order and first degree can always be written as

$$M+N\frac{dy}{dx}=0.$$

For convenience this equation is generally written as

$$M\,dx+N\,dy=0.$$

The following methods help us in solving such differential equations easily :

1. Integration by Inspection,
2. Separation of variables,
3. Homogeneous equations,
4. Linear equations.

2·2. Integration by Inspection. Some equations are easily integrated by mere inspection. This is generally the case when the equations are exact. For example, $y^2\,dx+2xy\,dy$ is the exact differential of $y^2\,x$ and hence the solution of the differential equation

$$y^2\,dx+2xy\,dy=0$$

is at once written by inspection as

$$y^2\,x=c,$$

where c is an arbitrary constant.

Ex 1. *Solve the differential equation*

$$\frac{dy}{dx}+\frac{ax+hy+g}{hx+by+f}=0.$$

We have $\quad (ax+hy+g)\,dx+(hx+by+f)\,dy=0$

or $\quad axdx+bydy+h(ydx+xdy)+gdx+fdy=0.$

Integrating it we get

$$\tfrac{1}{2}ax^2+\tfrac{1}{2}by^2+hxy+gx+fy=\text{constant}$$

or $\quad ax^2+by^2+2hxy+2gx+2fy=c,$

where c is an arbitrary constant.

2·21. Integrating Factors by Inspection. Sometimes the equation cannot be integrated directly as it stands but it becomes inte-

grable (*by* inspection) when it is multiplied by some function of x and y. The function which multiplies the equation to make it exact is called **Integrating factor.** As soon as the equation is multiplied by the integrating factor, the solution of the equation can be found out by inspection.

To be able to find out the integrating factor for a given equation by inspection, we should be able to recognize the differentials of some simpler functions. Typical examples of these are

$$xdy+ydx, \frac{xdy-ydx}{x^2}, \frac{xdy-ydx}{y^2}, \frac{xdy-ydx}{xy}, \frac{ydx-xdy}{x^2+y^2}$$

$$\frac{xdy-ydx}{x^2+y^2}, \frac{2xydy-y^2dx}{x^2}, \frac{y^2dx+2xydx}{x^2y^4}, \frac{xdy+ydx}{\sqrt{1-x^2y^2}},$$

which are easily seen to be the differentials of

xy, y/x, $-x/y$, $\log(y/x)$, $\tan^{-1}(x/y)$, $\tan^{-1}(y/x)$, y^2/x, $-1/xy^2$, $\sin^{-1} xy$,

respectively.

Ex. 2. *Solve* : $\quad ydx-xdy=0.$

The equation, if divided by the integrating factor $1/y^2$ becomes

$$\frac{ydx-xdy}{y^2}=0,$$

which is integrated by inspection to give

$$x/y=\text{constant}.$$

Ex. 3. *Solve* : $\quad \cot y\, dx-\tan x\, dy=0.$

Multiplying by $\sin y \cos x$, we get

$$\cos x \cos y\, dx-\sin x \sin y\, dy=0$$

which is integrated easily to give

$$\sin x \cos y=c,$$

where c is an arbitrary constant.

Ex. 4. *Solve* : $\quad (x^3+xy^2+a^2y)\, dx+(y^3+yx^2-a^2x)\, dy=0.$

We have $\quad (x^2+y^2)\,(xdx+ydy)+a^2\,(ydx-xdy)=0$

or $\quad xdx+ydy+a^2\,\dfrac{ydx-xdy}{x^2+y^2}=0.$

Integrating this we get

$$\tfrac{1}{2}\,(x^2+y^2)+a^2 \tan^{-1}(x/y)=c,$$

where c is an arbitrary constant.

Ex. 5. *Solve* : $\quad (x+2y^3)\,\dfrac{dy}{dx}=y.$

We have $\quad 2y^3\, dy=ydx-xdy,$

or $\quad 2y\, dy=(ydx-xdy)/y^2.$

Integrating this we get

$$y^2+c=x/y,$$

where c is an arbitrary constant.

EXERCISE II (A)

1. $(y+x)\,dx+x\,dy=0.$

2. $y\,(1+xy)\,dx-x\,dy=0.$

3. $\sin x\,\dfrac{dy}{dx}-y\cos x+y^2=0.$

4. $(x+y)dy+(y-x)dx=0.$

5. $x\,dx+y\,dy+\dfrac{x dy-y dx}{x^2+y^2}=0.$

6. $(x^3+3xy^2)\,dx+(y^3+3x^2y)\,dy=0.$

7. $x\,\dfrac{dy}{dx}+y=y^2\log x.$

8. $(1-x^2)\dfrac{dy}{dx}-2xy=x-x^3.$

9. $x\,dy-y\,dx-\cos(1/x)\,dx=0.$

10. $\dfrac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)^{1/2}}+\dfrac{z\,dx-x dz}{x^2+z^2}+3ax^2\,dx+2by\,dy+cdz=0.$

11. $x\,dx+y\,dy+\dfrac{x\,dy-y\,dx}{x^2+y^2}=0.$

12. $x\,dx+y\,dy=m\,(x\,dy-y\,dx).$

[Hint. Divide both sides x^2+y^2 and integrate].

2·3. Separation of Variables.

When the equation $M\,dx+N\,dy=0$ can be put in the form

$$f_1(x)\,dx+f_2(y)\,dy=0,$$

then it can be easily solved by integrating each term separately.

The solution of the equation given above is

$$\int f_1(x)\,dx+\int f_2(y)\,dy=c,$$

where c is an arbitrary constant.

Ex. 6. **Solve* $\sec^2 x \tan y\,dx+\sec^2 y \tan x\,dy=0.$

We have $\dfrac{\sec^2 x}{\tan x}\,dx=-\dfrac{\sec^2 y}{\tan y}\,dy,$

in which case the variables x and y have been separated. Integrating we get

$$\log\tan x=-\log\tan y+\text{constant}$$

i.e. $\tan x.\tan y=\text{constant}.$

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*This can be integrated easily by inspection also as it is an exact equation, the left hand side being the differential of $\tan x \tan y$.

Ex. 7. *Find the foci of the curve which satisfies the differential equation*

$$(1+y^2)\, dx - xy\, dy = 0,$$

and passes through the point (1, 0).

We have $$\frac{dx}{x} - \frac{y dy}{1+y^2} = 0.$$

Integrating it we get

$$\log x - \tfrac{1}{2} \log (1+y^2) = \log c$$

or $$x = c\sqrt{1+y^2}$$

If the curve passes throught (1, 0), we must have $c=1$, hence the curve is

$$x = \sqrt{1+y^2} \text{ or } x^2 - y^2 = 1.$$

It is a rectangular hyperbola, and its foci are evidently $(\pm\sqrt{2}, 0)$.

EXERCISE II (B)

1. $x \cos^2 y\, dx = y \cos^2 x\, dy$
2. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
3. $x^2 \dfrac{dy}{dx} + y = 1$
4. $ydx + (1+x^2) \tan^{-1} x\, dy = 0$
5. $(xy^2 + x)\, dx + (y x^2 + y)\, dy = 0$
6. $\dfrac{dy}{dx} = e^{x+y} + x^2 e^y$
7. $(3+2 \sin x + \cos x)\, dy = (1+2 \sin y + \cos y)\, dx$
8. $x^{-1} \cos^2 y\, dy + y^{-1} \cos^2 x\, dx = 0$
9. $(e^x+1)\, y\, dy = (y+1)\, e^x\, dx$
10. $\operatorname{cosec} x \log y\, dy + x^2 y^2\, dx = 0$
11. $\dfrac{dy}{dx} = \dfrac{\sin x + x \cos x}{y\,(2 \log y + 1)}$
12. $\cos y \log (\sec x + \tan x)\, dx = \cos x \log (\sec y + \tan y)\, dy$
13. $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$
14. $(\sin y + y \cos y)\, dy - (2 \log x + 1)\, x\, dx = 0$
15. $3e^x \tan y\, dx + (1-e^x) \sec^2 y\, dy = 0$
16. $y - x \dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx} \right).$

2·4. Homogeneous Equations. This case includes equations which can be expressed in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

In such cases substituting $y=vx, \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ enables us to solve the equation.

Ex. 8. *Solve :* $y^2\,dx+(xy+x^2)\,dy=0.$

Since this is easily seen to be the case of homogeneous equation, put $y=vx$ and $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ to give

$$v+x\frac{dv}{dx}=-\frac{v^2}{1+v}$$

or $$x\frac{dv}{dx}=-\frac{v+2v^2}{1+v}$$

or $$-\frac{dx}{x}=\frac{(1+v)\,dv}{v+2v^2}$$

$\therefore$ $$\log x=\int\frac{(1+v)\,dv}{v\,(2v+1)}$$

$$=\int\frac{dv}{v}-\int\frac{dv}{2v+1}$$

$$=\log v-\tfrac{1}{2}\log(2v+1)+\text{constant}$$

or $$2\log x+2\log v-\log(2v+1)=\log c,$$

where c is an arbitrary constant.

Hence we get $x^2\,v^2/(2v+1)=c.$

Since $v=y/x$, this reduces to

$$xy^2=c\,(2y+x),$$

which is the required solution.

2·41. Non-homogeneous equations of the first degree in x and y.

These can be put in the form

$$\frac{dy}{dx}=\frac{ax+by+c}{a'\,x+b'\,y+c'} \qquad \text{...(1)}$$

In this the substitution $x=X+h$ and $y=Y+k$ where h, k are constants, gives

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y} \qquad \text{...(2)}$$

provided that $ah+bk+c=0,$
and $a'h+b'k+c'=0,$ $\quad$ ...(A)

Relations (A) determine the constants h and k. With these values of h and k, (1) transforms to the form (2), which is the homogeneous form and can be integrated by the method given before.

Ex. 9. *Solve* : $\frac{dy}{dx}=\frac{6x-2y-7}{2x+3y-6}$ ......(1)

Putting $x=X+h$, $y=Y+k$, so that $dx=dX$, $dy=dY$, we get

$$\frac{dY}{dX}=\frac{6X-2Y}{2X+3Y} \qquad \text{......(2)}$$

provided that $\quad 6h-2k-7=0,$
and $\quad 2h+3k-6=0,$
which give $h=3/2$ and $k=1$.

In (2) putting $Y=vX$ and $\frac{dY}{dX}=v+X\frac{dv}{dX}$, we get on simplification.

$$-\frac{dX}{X}=\frac{6v+4}{2(3v^2+4v-6)}\,dv,$$

which on integration gives

$$X^2(3v^2+4v-6)=c,$$

where c is an arbitrary constant.

Substituting the values of v, we get

$$3Y^2+4YX-6X^2=c,$$

in which putting $Y=y-k=y-1$ and $X=x-h=x-3/2$, we get

$$3(y-1)^2+4(y-1)(x-3/2)-6(x-3/2)^2=c$$

or $\quad 3y^2+4xy-6x^2-12y+14x=k,$

where k is an arbitrary constant.

2·42. Special Case. Sometimes the differential equation can be brought to the form

$$\frac{dy}{dx}=\frac{ax+by+c}{r(ax+by)+c'}$$

where r is any number.

In such cases the substitution $ax+by=t$, $a+b\frac{dy}{dx}=\frac{dt}{dx}$, transforms the differential equation to a form which can be solved easily.

Ex. 10. *Solve* : $(2x+y+1)\,dx+(4x+2y-1)\,dy=0.$

We have $\frac{dy}{dx}=-\frac{2x+y+1}{2(2x+y)-1}$.

Putting $2x+y=t$, we get

$$\frac{dt}{dx}-2=-\frac{t+1}{2t-1}$$

$$\therefore \quad \frac{dt}{dx}=\frac{3\ (t-1)}{2t-1}$$

This easily gives

$$x=\int \frac{2t-1}{3\ (t-1)}\ dt$$

$$=\int \frac{2\ (t-1)+1}{3\ (t-1)}\ dt$$

$$=\frac{2}{3}\int dt+\frac{1}{3}\int \frac{dt}{t-1}$$

$$\therefore \quad 3x=2t+\log\ (t-1)+c$$

$$=2\ (2x+y)+\log\ (2x+y-1)+c$$

or $$x+2y+\log\ (2x+y-1)+c=0$$

EXERCISE II (C)

Solve :

1. $(x+y-1)\ dy=(x+y+1)\ dx$
2. $(2x+2y+1)\ dy=(x+y+1)\ dx$
3. $(2x+3y-5)\ dy+(2x+3y-1)\ dx=0$
4. $(x^2+y^2)\ dy=(x^2+xy)\ dx$
5. $[\ x\cos\ (y/x)+y\sin\ (y/x)\ ]\ y\ dx$
$-[\ y\sin\ (y/x)-x\cos\ (y/x)\ ]\ x\ dy=0$
6. $(x^2-y^2)\ dx+2xy\ dy=0$ 7. $\dfrac{dy}{dx}=\dfrac{y}{x}+\tan\ \dfrac{y}{x}$
8. $(2x-2y+5)\ dy-(x-y+3)\ dx=0$
9. $(x+y+1)\ dx-(2x+2y+1)\ dy=0$
10. $y^2=(xy-x^2)\ \dfrac{dy}{dx}$
11. $x\sin\ (y/x)\ dy=[\ y\sin\ (y/x)-x\ ]\ dx$
12. $(x^2+y^2)\ dy=xy\ dx$ 13. $x^2\ dy+y\ (x+y)\ dx=0$
14. $2\ \dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{y^2}{x^2}$
15. $(6x-5y+4)\ dy+(y-2x-1)\ dx=0$
16. $(x-3y+4)\ dy+(7y-5x)\ dx=0$
17. $(2x+4y+3)\ dy=(2y+x+1)\ dx$

18. $xdy-ydx=\sqrt{x^2+y^2}\,dx.$
19. $x\,(x^2+3y^2)\,dx+y\,(y^2+3x^2)\,dy=0$
20. $(x^2+y^2)\,dx-2xy\,dy=0$
21. $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$ 22. $(x-y)\,dy=(x+y+1)\,dx.$
23. $(x-y-2)\,dx-(2x-2y-3)\,dy=0.$

2·5. Linear Equations. *An equation of the form*

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are functions of x (or constants) is called a **linear equation** *of the first order.*

Suppose R is an integrating factor, then the left hand side of

$$R\frac{dy}{dx}+RPy=RQ$$

is the differential coefficient of some product. Since the term $R\dfrac{dy}{dx}$ can only be derived by differentiating Ry, we put

$$R\frac{dy}{dx}+RPy=\frac{d}{dx}(Ry)=R\frac{dy}{dx}+y\frac{dR}{dx}$$

$$\therefore \qquad RP=\frac{dR}{dx}$$

or $$\log R=\int P\,dx, \text{ whence } R=e^{\int P\,dx}.$$

Hence we have determined the integrating factor, R, and so we have the rule :

To solve $\dfrac{dy}{dx}+Py=Q$, multiply both sides by $e^{\int P\,dx}$, which is an integrating factor and then we get the integral as

$$y.\,e^{\int P\,dx}=\int Q\,e^{\int P\,dx}\,dx$$

Ex. 11. *Solve :* $\cos x\dfrac{dy}{dx}+y\sin x=1.$

We have to solve

$$\frac{dy}{dx}+y\tan x=\sec x. \qquad \text{......(1)}$$

Since $P=\tan x$, we have

$$\int P\,dx=\int \tan x\,dx=\log \sec x,$$

$\therefore$ The integrating factor is

$$e^{\log \sec x}=\sec x$$

Multiplying (1) by $\sec x$, we have

$$\sec x\,\frac{dy}{dx}+y\sec x\tan x=\sec^2 x$$

$$\therefore\quad y\sec x=\int \sec^2 x\,dx=\tan x+c,$$

where c is an arbitrary constant.

Ex. 12. *Solve :* $\dfrac{dy}{dx}+2xy=e^{-x^2}$

Here $P=2x$, and so

$$\int P\,dx=\int 2x\,dx=x^2,$$

and the integrating factor is e^{x^2}.

Multiplying the equation by the integrating factor we get

$$e^{x^2}\frac{dy}{dx}+2x\,e^{x^2}\,y=1,$$

which gives $\qquad e^{x^2}.\,y=\int dx=x+c,$

where c is an arbitrary constant.

Ex. 13. *Solve :* $(x+2y^3)\dfrac{dy}{dx}=y.$

We have $\dfrac{dx}{dy}-\dfrac{1}{y}x=2y^2.$

This is a linear equation if we take x as the dependent and y as the independent variable. Hence integrating factor is

$$e^{\int P\,dy}=e^{-\int (1/y)\,dy}=\frac{1}{y}$$

Thus we have $\dfrac{1}{y}\left(\dfrac{dx}{dy}-\dfrac{x}{y}\right)=\dfrac{1}{y}.\,2y^3,$

which gives the solution as

$$x\,(1/y)=y^2+c$$

or $$x=y\,(y^2+c).$$

Aliter. The equation can be put as

$$2y\ dy=(x\ dy-y\ dx)/y^2,$$

which easily gives the solution by inspection.

2·6. Equations reducible to the linear form.

Some equations can, on suitable substitutions, be reduced to the linear form, and hence can be solved easily. This method is illustrated below.

Ex. 14. **Solve* $\frac{dy}{dx}+Py=Qy^n$, *where P and Q are functions of x and n is any number.*

The equation can be written as

$$y^{-n}\frac{dy}{dx}+P\ y^{1-n}=Q.$$

Put $y^{1-n}=v$, so that $\frac{dv}{dx}=(1-n)\ y^{-n}\ \frac{dy}{dx}$.

The equation transforms to

$$\frac{dv}{dx}+(1-n)\ Pv=\ (1-n)\ Q,$$

which being linear in v can be solved easily by the method given for linear equations.

Ex. 15. *Solve :* $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y.$

The equation can be written as

$$\sec^2 y\ \frac{dy}{dx}+2x\tan y=x^3.$$

Putting $\tan y=v$, so that $\sec^2 y\ \frac{dy}{dx}=\frac{dv}{dx}$,

we get $$\frac{dv}{dx}+2xv=x^3,$$

for which $P=2x$ and so $\int P\ dx=\int 2x\ dx=x^2$

$\therefore$ Integrating factor is e^{x^2}.

The equation in v becomes

$$e^{x^2}\ \frac{dv}{dx}+2xv\ e^{x^2}=\int x^3\ e^{x^2}\ dx.$$

---

*The equation is Bernoulli's Equation and was studied by him in 1695.

Integrating this we get

$$e^{x^2}.\, v=\tfrac{1}{2} \int t\, e^t\, dt, \qquad \text{where } t=x^2$$

$$=\tfrac{1}{2}\, [\, t\, e^t-\int e^t\, dt\,]=\tfrac{1}{2}\, e^t\, (t-1)+c$$

$$\therefore \quad e^{x^2} \tan y=\tfrac{1}{2}\, e^{x^2}\, (x^2-1)+c,$$

where c is an arbitrary constant.

EXERCISE II (D)

Solve the following equations :—

1. $\dfrac{dy}{dx}+y \cot x=2 \cos x.$

2. $\cos^2 x \dfrac{dy}{dx}+y=\tan x.$

3. $x \cos x \dfrac{dy}{dx}+y\,(x \sin x+\cos x)=1.$

4. $(y-x \sin x^2)\, dx+x\, dy=0.$

5. $x \log x \dfrac{dy}{dx}+y=2 \log x.$

6. $\sin x \cos x \dfrac{dy}{dx}=y+\sin x.$

7. $(1+x+xy^2)\, dy+(y+y^3)\, dx=0.$

8. $y^2\, dx+\left(x-\dfrac{1}{y}\right) dy=0.$

9. $\dfrac{dy}{dx}+3x^2y=x^5\, e^{x^3}.$

10. $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)\, e^x \sec y.$

11. $\dfrac{dy}{dx}+\dfrac{1-2x}{x^2}\, y=1.$

12. $\dfrac{dy}{dx}+\dfrac{2}{x}\, y=\sin x$

13. $(1+y^2)dx=(\tan^{-1}y-x)dy.$

14. $(1+y+x^2y)\, dx+(x+x^3)\, dy=0.$

15. $\dfrac{dy}{dx}+\dfrac{x}{1+x^2}\, y=\dfrac{1}{2x\,(1+x^2)}$

16. $\dfrac{dy}{dx}+\dfrac{1}{x} \tan y=\dfrac{1}{x^2} \tan y \sin y.$ (Put $y=\sin^{-1} t$)

17. $\dfrac{dy}{dx}+\dfrac{y}{x} \log y=\dfrac{y}{x^2}\, (\log y)^2.$ (Put $y=e^t$).

18. $\dfrac{dy}{dx}+x=x\,e^{(n-1)y}$. (Put $y=\log t$).

19. $y\,(2xy+e^x)\,dx-e^x\,dy=0$.

20. $2\dfrac{dy}{dx}-y\sec x=y^3\tan x$. 21. $\dfrac{dy}{dx}+y\cos x=y^n\sin 2x$.

2·7. Further Remarks on Exact Equations. In order that the ordinary differential equation $Mdx+Ndy=0$ should be exact, there must exist a function $u\,(x, y)$ such that $du=Mdx+Ndy$.

Theorem. **The necessary and sufficient condition for the ordinary differential equation $Mdx+Ndy=0$ to be exact is**

$$\frac{\partial M}{dy}=\frac{\partial N}{\partial x}.$$

The condition is necessary.

$$\text{Now}\quad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}\,dy.$$

Further, since the equation is exact, we have

$$du=Mdx+Ndy$$

$$\therefore\quad M=\frac{\partial u}{\partial x}\text{ and }N=\frac{\partial u}{\partial y}$$

$$\text{or}\quad \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}\text{ and }\frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

$$\therefore\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

The condition is sufficient. We have to show that if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\text{, then }Mdx+Ndy=du.$$

$$\text{Let }\int Mdx=P.\quad\therefore\quad M=\frac{\partial P}{\partial x}$$

$$\text{or}\quad \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}=\frac{\partial^2 P}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$$

$$\therefore\quad N=\frac{\partial P}{\partial y}+f\,(y).$$

Using these values of M and N, we get

$$Mdx+Ndy=\frac{\partial P}{\partial x}\cdot dx+\frac{\partial P}{\partial y}dy+f(y)dy$$
$$=d\,[\,P+F\,(y)\,],$$

where $d\,[\,F\,(y)\,]=f\,(y)\,dy$.

Writing $P+F\,(y)=u\,(x,y)$, we get

$$Mdx+Ndy=du.$$

Hence the condition is also sufficient.

2·8. Method for solving an exact equation. To solve the exact equation $Mdx+Ndy=0$, it is necessary to find the function $u\,(x,y)$. It is often possible to arrange the terms in groups each of which is an exact differential and hence $u\,(x,y)$ can be obtained by inspection only [Refer Art. 2·2].

If this cannot be done, we first ascertain with the help of the condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ whether the equation is exact or not. If the equation is exact, the value of $u\,(x,y)$ can be found with the help of the relation

$$\frac{\partial u}{\partial x}=M.$$

Partial integration of this gives

$$u=\int Mdx+f\,(y),$$

the constant being possibly a function of y since y is treated as constant during the integration.

The function $f\,(y)$ can be found by equating the total differential of u to $Mdx+Ndy$.

Since all the terms of $u\,(x,y)$ which contain x must appear in $\int Mdx$, the differential of this integral with respect to y must have all terms of Ndy which contain x; hence we have the following rule for solving the exact equation $Mdx+Ndy=0$:—

"*First integrate the terms in Mdx treating y as constant, then integrate those terms of Ndy which do not contain x, and equate the sum of these integrals to a constant.*"

Ex. 16. *Solve* : $(a^2-2xy-y^2)\,dx-(x+y)^2\,dy=0$.

Here $\frac{\partial M}{dy}=\frac{\partial N}{dx}=-2\,(x+y)$,

and hence the equation is exact,

$$\int Mdx=a^2\,x-x^2\,y-xy^2,$$

and $-y^2\,dy$ is the only term in Ndy free from x, and so the solution is

$$a^2x-x^2y-xy^2-y^3/3=c.$$

EXERCISE II (E)

Solve the following differential equations :

1. $xdx+ydy=\dfrac{a^2\,(xdy-ydx)}{x^2+y^2}$.

2. $(1+4xy+2y^2)\,dx+(1+4xy+2x^2)\,dy=0$.

2·9. Integrating factors. We proceed to show that the number of integrating factors for an equation $M\,dx+N\,dy=0$ is infinite.

Let μ be an integrating factor, so that

$$\mu\,(M\,dx+N\,dy)=du$$

and hence $u=c$ is a solution.

If $f\,(u)$ is any function of u, we have

$$\mu\,f\,(u)\;[\;M\,dx+N\,dy\;]=f\,(u)\;du.$$

The right hand expression is an exact differential since $f(u)\,du$ can be easily integrated to give $F\,(u)$. Thus a solution of the equation is $F\,(u)=c$. Hence we see that $\mu\;f\,(u)$ is also an integrating factor of $M\,dx+N\,dy=0$, and since $f\,(u)$ is an arbitrary function of u, the number of integrating factors is infinite.

Rules for finding out integrating factors. We have already seen that sometimes an integrating factor can be found out by inspection only. When it is not possible to get an integrating factor by inspection, some rules enable us to find it. These rules are given below.

Particular case. When $Mx\pm Ny=0$ the equation

$$Mdx+Ndy=0$$

becomes $$\frac{dy}{dx}=-\frac{M}{N}=\pm\,\frac{y}{x},$$

which can be integrated easily, and no integrating factor is necessary.

Rule I. **When $Mx+Ny\neq 0$ and the equation is homogeneous.**

$\dfrac{1}{Mx+Ny}$ **is an integrating factor of** $Mdx+Ndy=0$.

We have

$$M\,dx+N\,dy$$

$$=\tfrac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$=\tfrac{1}{2}\,\{(Mx+Ny)\;d\,[\;\log xy\;]+(Mx-Ny)\;d\;[\log\,(x/y)]\}\qquad(A)$$

Since $Mx+Ny\neq 0$, we have

$$\frac{M\,dx+N\,dy}{Mx+Ny}=\frac{1}{2}\,d\,[\,\log xy\,]+\frac{1}{2}\;\frac{Mx-Ny}{Mx+Ny}\,d\,[\,\log (x/y)\,]$$

Now $M\,dx+N\,dy$ is homogeneous and hence $\dfrac{Mx-Ny}{Mx+Ny}$ is homogeneous and equal to a function of (x/y), say $f\,(x/y)$, so that

$$\frac{M\,dx+N\,dy}{Mx+Ny}\;\tfrac{1}{2}\,d\,[\log (xy)]+\tfrac{1}{2}\,f\,(x/y)\;d\,[\log (x/y)]$$

$$=\tfrac{1}{2}\,d\,[\log (xy)]+\tfrac{1}{2}\,F\{\log (x/y)\}.\;d\,[\log (x/y)],$$

$$\text{since}\quad f\,(x/y)=f\,[e^{\log (x/y)}\,]=F\,[\log (x/y)]$$

The right hand side of this relation is an exact differential.

Hence $\dfrac{1}{Mx+Ny}$ is an integrating factor.

Rule II. **When $Mx-Ny\neq 0$ and the equation can be written as**

$$[\,f\,(xy)\,]\;y\,dx+F\,[\,(xy)\,]\;x\,dy=0.$$

$\dfrac{1}{Mx+Ny}$ **is an integrating factor.**

Since $Mx-Ny\neq 0$, dividing (A) of Rule I by $Mx-Ny$, we get

$$\frac{M\,dx+N\,dy}{Mx-Ny}=\frac{1}{2}\;\frac{Mx+Ny}{Mx-Ny}\,d\,[\log xy]+\tfrac{1}{2}\,d\,[\log (x/y)].$$

Since $M\equiv[\,f\,(xy)\,]\,y$ and $N\equiv[\,F\,(xy)\,]\,x$, we have

$$\frac{M\,dx+N\,dy}{Mx-Ny}=\frac{f\,(xy)+F\,(xy)}{f\,(xy)-F\,(xy)}\,d\,[\log (xy)]+\tfrac{1}{2}d\,[\log (x/y)]$$
$$=\tfrac{1}{2}\,\phi\,(xy)\;d\,[\log (xy)\,]+\tfrac{1}{2}\,d\,[\,\log (x/y)]$$
$$=\tfrac{1}{2}\,\psi\,[\,\log (xy)\,].\;d\,[\,\log (xy)\,]$$
$$+\tfrac{1}{2}\,d\,[\,\log (x/y)\,],$$

$$\text{since}\;\phi\,(xy)=\phi\,[e^{\log (xy)}=\psi\,[\,\log (xy)\,].$$

The right hand expression of the above relation is an exact differential and hence $\dfrac{1}{Mx-Ny}$ is an integrating factor.

Rule III. **When $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then**

$$e^{\int f(x)\,dx}$$

is an integrating factor.

Multiplication of $M\,dx+N\,dy=0$ by $e^{\int f(x)\,dx}$ gives say $M_1\,dx+N_1\,dy=0$, where

$$M_1=M\,e^{\int f(x)\,dx} \text{ and } N_1=N\,e^{\int f(x)\,dx}$$

Now $$\frac{\partial N_1}{\partial x}=e^{\int f(x)\,dx}\,\frac{\partial N}{\partial x}+N.f(x)\,e^{\int f(x)\,dx}$$

$$=e^{\int f(x)\,dx}\,\frac{\partial N}{dx}+\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]e^{\int f(x)\,dx}$$

$$=e^{\int f(x)\,dx}\,\frac{\partial M}{\partial y}=\frac{\partial M_1}{\partial y},$$

so that $M_1\,dx+N_1\,dy=0$ is also exact and hence $e^{\int f(x)\,dx}$ is an integrating factor.

Aliter. Let μ be an integrating factor of $M\,dx+N\,dy=0$. Multiplying by μ, we get

$$(\mu M)\,dx+(\mu N)\,dy=0.$$

This must be exact and hence the condition

$$\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$$

or $$M\frac{\partial\mu}{\partial y}-N\frac{\partial\mu}{\partial x}+\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=0$$

Now suppose μ is a function of x only, so that $\frac{\partial\mu}{\partial y}=0$, then

$$\frac{\partial\mu}{\mu}=\frac{1}{N}\left(\frac{\partial M}{dy}-\frac{\partial N}{\partial x}\right)dx \tag{A}$$

Now since μ is a function of x alone, the right hand side of A must also be a function of x only. Let us take

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x),$$

so that (A) becomes

$$\frac{\partial\mu}{\mu}=f(x)\,dx,$$

giving $\mu=e^{\int f(x)\,dx}$. Hence the result.

Rule IV. **When $\frac{1}{M}\left(\frac{\partial N}{dx}-\frac{\partial M}{\partial y}\right)$ is a function of y alone, say $\phi(y)$, then $e^{\int \phi(y)\,dy}$ is an integrating factor.**

This can be proved in the same way as Rule III.

Rule V. **If the equation is of the form**

$$x^a y^b (my\,dx+nx\,dy)=0,$$

then $x^{km-a-1} y^{kn-b-1}$, where k has any value, is an integrating factor.

Assume that $x^p y^q$ is an integrating factor of

$$x^a y^b (my\,dx+nx\,dy)=0.$$

Now $x^{p+a} y^{q+b} (my\,dx+nx\,dy)$ is exact if

$$m(q+b+1)=n(p+a+1) \text{ i. e. } \frac{q+b+1}{n}=\frac{p+a+1}{m}=k,$$

where k is any number whatever.

$$\therefore \quad p=km-a-1, \text{ and } q=kn-b-1.$$

The integral of the exact differential

$$x^{km-1} y^{kn-1} (my\,dx+nx\,dy),$$

is easily seen to be $(1/k)\,x^{km} y^{kn}$ if $k\neq 0$.

Thus we see that $x^{km-a-1} y^{kn-b-1}$, where k is any number whatever, is an integrating factor of

$$x^a y^b (my\,dx+nx\,dy)=0.$$

When an equation can be put in the form

$$x^a y^b (my\,dx+nx\,dy)+x^c y^d (m_1\,ydx+n_1\,xdy)=0,$$

we can easly find an integrating factor.

An integrating factor of first term is $x^{km-a-1} y^{kn-b-1}$ and that of the second term is $x^{k_1m_1-c-1}\ y^{k_1n_1-d-1}$, where k, k_1 have any values. These two factors are identical if

$$km-a-1=k_1\,m_1-c-1 \text{ and } kn-b-1=k_1\,n_1-d-1.$$

These easily determine k and k_1. [See solved example 21]

2·10. Illustrations. Some examples to illustrate the application of these rules are given below.

Ex. 17. *Solve :* $x^2y\,dx-(x^3+y^3)\,dy=0.$

Here $Mx+Ny=x^3y-x^3y-y^4=-y^4$, hence $-1/y^4$ is an integrating factor.

$\therefore$ The equation becomes

$$-\frac{x^2}{y^3}\,dx+\frac{x^3+y^3}{y^4}\,dy=0,$$

or
$$\frac{dy}{y}=\frac{x^2}{y^3}\,dx-\frac{x^3}{y^4}\,dy$$

or
$$\log y=\frac{1}{3}\,\frac{x^3}{y^3}+k.$$

$\therefore\quad y=c\,e^{x^3/3y^3}$, where c is an arbitrary constant.

Ex. 18. *Solve :*

$$(xy\sin xy+\cos xy)\,ydx+(xy\sin xy-\cos xy)\,xdy=0.$$

Here $Mx-Ny=2xy\cos xy$, and we see that in this case Rule II can be applied considering the form of the equation.

The integrating factor by Rule II is $1/2xy\cos xy$, and hence the equation becomes

$$\tan xy.\,(ydx+xdy)+\frac{dx}{x}-\frac{dy}{y}=0$$

or $\log\sec xy+\log x-\log y=\log c$

$$\therefore\quad\frac{x}{y}\sec xy=c,$$

where c is an arbitrary constant.

Ex. 19. *Solve :* $(y+\frac{1}{3}y^3+\frac{1}{2}x^2)\,dx+\frac{1}{4}(x+xy^2)\,dy=0.$

We have $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{3}{4}(1+y^2)\div\left[\frac{x}{4}(1+y^2)\right]$

hence $I.F.=e^{\int(3/x)\,dx}=x^3.$

The equation becomes

$$4x^3ydx+x^4dy+\tfrac{4}{3}x^3y^3dx+x^4y^2dy+2x^5dx=0$$

or $x^4y+\frac{1}{3}x^4y^3+\frac{1}{3}x^6=k.$

$\therefore$ The solution is given by $3x^4y+x^4y^3+x^6=c$,

where c is an arbitrary constant.

Ex. 20. *Solve* : $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$.

We have

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-2\ (2x+3x^2y^3)\div[\,y\ (2x+3x^2y^3)\ ]=-2/y$$

Hence $IF=e^{-\int (2/y)\,dy}=1/y^2$.

The equation becomes

$$3x^2y^2dx+2x^3ydy+\frac{2x}{y}\,dx-\frac{x^2}{y^2}\,dy=0.$$

$\therefore$ The solution is $x^3y^2+x^2/y=c$,
where c is an arbitrary constant.

Ex. 21. *Solve* : $(y^3-2yx^2)\,dx+(2xy^2-x^3)\,dy=0$.

The equation can be written as

$$y^2\ (ydx+2xdy)-x^2\ (2ydx+xdy)=0.$$

In the case of first term $a=0$, $b=2$, $m=1$, $n=2$ and hence the integrating factor according to Rule V is $x^{k-1}\,y^{2k-3}$, where k has any value.

Similarly for the second term $a=2$, $b=0$, $m=2$, $n=1$ and hence $x^{2k'-3}\,y^{k'-1}$ is an integrating factor.

These factors will be the same if

$$k-1=2k'-3 \text{ and } 2k-3=k'-1,$$

which give $k=k'=2$.

Hence xy is the common integrating factor for both the terms.

The equation becomes

$$xy^3\ (ydx+2xdy)-x^3y\ (2ydx+xdy)=0$$

or $$xy^4\,dx+2x^2y^3dy-(2x^3y^2dx+x^4y\,dy)=0$$

or $$\tfrac{1}{2}\,x^2y^4-\tfrac{1}{2}\,x^4y^2=\text{constant}.$$

Hence the solution is

$$x^2y^2\ (y^2-x^2)=c,$$

where c is an arbitrary constant.

EXERCISE II (F)

Solve the following equations :

1. $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0$.
2. (*a*) $(x^4y^4+x^2y^2+xy)\,y\,dx+(x^4y^4-x^2y^2+xy)\,x\,dy=0$.
   (*b*) $y\,(xy+2x^2y^2)\,dx+x\,(xy-x^2y^2)\,dy=0$.
3. $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0$.
4. (*a*) $(x^2+y^2)\,dx-2xy\,dy=0$.
   (*b*) $(20x^2+8xy+4y^2+3y^3)\,y\,dx$
   $+4\,(x^2+xy+y^2+y^3)\,x\,dy=0$.

Qr

5. (a) $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$
(b) $(2y\,dx+3x\,dy)+2xy\,(3y\,dx+4x\,dy)=0.$

Miscellaneous Examples on Chapter II

Solve the following differential equations :

1. $\dfrac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left(\dfrac{a^2-x^2-y^2}{x^2+y^2}\right)}$

[**Hint.** Change to Polars.]

2. $\left(\dfrac{x+y-a}{x+y-b}\right)\dfrac{dy}{dx}=\dfrac{x+y+a}{x+y+b}.$

3. $(x-y)^2\dfrac{dy}{dx}=a^2.$

4. $(x+y)^2\dfrac{dy}{dx}=a^2.$

5. $\dfrac{dy}{dx}=(4x+y+1)^2.$ [Put $4x+y+1=t$]

6. $x\,Dy-y=x\sqrt{x^2+y^2}.$

7. $x\dfrac{dy}{dx}+y\log y=x\,y\,e^x.$

8. $1+y^2+\left(x-e^{-\tan^{-1}y}\right)\dfrac{dy}{dx}=0.$

9. $x\,\dfrac{dy}{dx}-y=\sqrt{x^2+y^2}.$

10. $(x^2+y^2-a^2)\,x\,dx+(x^2-y^2-b^2)\,y\,dy=0.$ [Eq^n is exact]

11. $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}.$

12. $x\,dx+y\,dy=m\,(x\,dy-y\,dx)$, by changing it into polars.

13. $y\,dx+(ax^2\,y^n-2x)\,dy=0.$

14. $y\,(2x^2y+e^x)\,dx-(e^x+y^3)\,dy=0.$

15 $\left(\dfrac{dx}{dt}\right)^2=k^2\,(1-e^{-2gx/k^2}).$

where k and g are constants, and $x=0$ when $t=0$.

16. $y\,dy+by^2\,dx=a\cos x\,dx.$

17. $\dfrac{dy}{dx}=e^{3x-2y}+x^2 e^{-2y}$.

18. $(x^2+y^2+x)\,dx-(2x^2+2y^2-y)\,dy=0$.

19. $(2y\,dx+3x\,dy)+2xy\,(3y\,dx+4x\,dy)=0$.

20. $\{y(1+1/x)+\cos y\}\,dx+\{x+\log x-x\sin y\}\,dy=0$.

21. $(2x+2y+3)\,dy=(x+y+1)\,dx$.

22. $\dfrac{dy}{dx}=\dfrac{x\,(2\log x+1)}{\sin y+y\cos y}$.

23. $\dfrac{ds}{dx}+x^2=x^2 e^{3s}$.

24. $\dfrac{dy}{dx}=e^{x-y}\,(e^x-e^y)$.

25. $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y)$.

26. $\dfrac{dy}{dx}+\dfrac{\tan y}{x}=\dfrac{1}{x^2}\tan y\sin y$.

27. $(x^2-ay)\,dx=(ax-y^2)\,dy$.

28. $y\,(2xy+e^x)\,dx-e^x\,dy=0$.

29. $y^2+x^2\dfrac{dy}{dx}=xy\dfrac{dy}{dx}$.

30. $\dfrac{dy}{dx}+\dfrac{y}{(1-x^2)^{3/2}}=\dfrac{x+\sqrt{1-x^2}}{(1-x^2)^2}$.

31. $y\,dx-x\,dy+(1+x^2)\,dx+x^2\sin y\,dy=0$.

32. $\sec^2 y\dfrac{dy}{dx}+2x\tan y=x^3$.

33. Solve the equation $\dfrac{dy}{dx}+\dfrac{ax+by+c}{bx+fy+e}=0$.

Hence show that if the tangent to a curve at any point (x, y) is known to be inclined at $\tan^{-1}\left\{\dfrac{ax+by+c}{bx+fy+e}\right\}$ to the x-axis, the curve must be a conic section.

CHAPTER III

ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

3·1. Definition. An ordinary *linear differential equation* of n^{th} order has the form*

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1} y}{dx^{n-1}}+\ldots\ldots+P_{n-1}\frac{dy}{dx}+P_n y=Q, \qquad \ldots\ldots(1)$$

where $P_1, P_2 \ldots\ldots\ldots, P_n$, are functions of x or constants and do not contain y or derivatives of y.

If $P_1, P_2 \ldots\ldots\ldots\ldots P_n$ are constants and Q is any function of x, the equations of the form given by (1) are called *ordinary differential equations with constant coefficients.*

In this chapter we will consider only the ordinary differential equations with constant coefficients. The following theorems are of great importance in the study of these linear equations.

Theorem 1. **If $y=f(x)$ is the general solution of**

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1} y}{dx^{n-1}}+\ldots\ldots+P_n y=0, \qquad \ldots\ldots(2)$$

and $y=\phi(x)$ is solution of equation (1), then

$$y=f(x)+\phi(x), \qquad \ldots\ldots(3)$$

is the general solution of

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1} y}{dx^{n-1}}+\ldots\ldots+P_{n-1}\frac{dy}{dx}+P_n y=Q. \qquad \ldots\ldots(1)$$

Substituting the value of y from (3) in (1) we get the left hand side of (1)

$$=\left(\frac{d^n f}{dx^n}+P_1\frac{d^{n-1} f}{dx^{n-1}}+\ldots\ldots+P_n f\right)$$

$$+\left(\frac{d^n \phi}{dx^n}+P_1\frac{d^{n-1}\phi}{dx^{n-1}}+\ldots\ldots+P_n \phi\right).$$

*If the coefficient of $\frac{d^n y}{dx^n}$ in not unity, it can always be made unity by dividing the equation by this coefficient. Thus (1) is the most general form.

Now, since $y=f(x)$ is a solution of (2), the first group of terms within the brackets is zero. Similarly the second group of terms is equal to Q since $y=\phi(x)$ is a solution of (1).

Hence (3) is a solution of (1).

This theorem enables us to divide the method of solving a linear equation into two parts.

First, we find the general solution of the equation (2) which we will denote, say, by

$$y=f_1(c_1, c_2 \ldots\ldots c_n, x).$$

Next, we find a solution of (1) which *does not contain an arbitrary constant*. Let us denote this solution, say, by

$$y=f_2(x).$$

Then $\qquad y=f_1(c_1, c_2 \ldots\ldots c_n, x)+f_2(x)$
gives evidently the general solution of equation (1).

Definitions. *The expression* $f_1(c_1, c_2 \ldots\ldots c_n, x)$ *is called the* **complementary function** *and* $f_2(x)$ *is called the* **particular integral.**

Theorem 2. **If** $y=y_1, y=y_2, \ldots\ldots, y=y_n$, **are solutions of the equation (2), then**

$$y=c_1 y_1+c_2 y_2+\ldots\ldots+c_n y_n, \tag{4}$$

where $c_1, c_2 \ldots\ldots, c_n$ **are arbitrary constants, is also a solution of the equation.**

Substituting the value of y from (4) in the equation (2), we get

$$\left[\frac{d^n y_1}{dx^n}+P_1\frac{d^{n-1} y_1}{dx^{n-1}}+\ldots\ldots+P_n y_1\right]$$

$$+c_2\left[\frac{d^n y_2}{dx^n}+P_1\frac{d^{n-1} y_2}{dx^{n-1}}+\ldots\ldots+P_n y_2\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+c_n\left[\frac{d^n y_n}{dx^n}+P_1\frac{d^{n-1} y_n}{dx^{n-1}}+\ldots\ldots+P_n y_n\right]=0.$$

Now, since $y=y_1, y=y_2 \ldots\ldots\ldots, y=y_n$ are solutions of equation (2), each expression in the brackets is zero. Hence (4) is a solution of (2).

This theorem enables us to find n solutions of the equation (2) as $y=y_1, y=y_2 \ldots\ldots, y=y_n$ and then express the solution as (4). Since the solution given by (4) contains n arbitrary constants, it is the general solution of equation (2) or the *complementary function* of equation (1).

Hence to find the general solution of (2) we must devise methods for finding n Independent integrals $y_1, y_2 \cdots\cdots y_n$.

Note :—(4) gives the general solution of (2), provided that $y_1, y_2 \ldots\ldots\ldots, y_n$ are *linearly independent* *i. e.* there does not exist a set of n constants $a_1, a_2 \ldots\ldots a_n$ at least one of which is different from zero, such that

$$a_1 y_1 + a_2 y_2 + \ldots\ldots\ldots + a_n y_n \equiv 0.$$

If these functions are not independent then some one of them, say y_n, can with the help of the identity given above be expressed in terms of the others as

$$y_n = -(a_1 y_1 + a_2 y_2 + \ldots\ldots + a_{n-1} y_{n-1})/a_n.$$

Hence it is clear that relation (4) can be written as

$$y = (c_1 - a_1 c_n/a_n) y_1 + \ldots\ldots + (c_{n-1} - a_{n-1} c_n/a_n) y_{n-1}$$

which contains only $(n-1)$ constants, and therefore is not the general solution.

3·2. The Symbolic Operator. The discussion of linear equations is facilitated by writing

$$D^r y \text{ for } \frac{d^r y}{dx^r}.$$

Thus we have $(D-m_1)\, y \equiv \dfrac{dy}{dx} - m_1 y$.

The notation $(D-m_1)(D-m_2)\, y$ is defined to denote that we operate on y with $(D-m_2)$ and then upon the result with $(D-m_1)$. Hence if m_1, m_2 are constants

$$\begin{aligned}(D-m_1)(D-m_2)\, y &= (D-m_1)(y' - m_2 y)\\ &= y'' - (m_1+m_2)\, y' + m_1 m_2 y\end{aligned}$$

where, as usual, $y'' = \dfrac{d^2y}{dx^2}$ and $y' = \dfrac{dy}{dx}$. In this case it is easily verified that

$$\begin{aligned}(D-m_1)(D-m_2)\, y &= (D-m_2)(D-m_1)\, y\\ &= [\, D^2 - (m_1+m_2)\, D + m_1 m_2 \,]\, y.\end{aligned}$$

Thus we see that if m_1, m_2 are constants, the value of $(D-m_1)(D-m_2)\, y$ is independent of the order in which the operational factors are used.

3·3. Solution of the Linear Equation with constant coefficients.

Throughout the remaining part of this chapter, the coefficients $P_1, P_2 \ldots\ldots\ldots, P_n$ will be taken to be constants.

Now, the homogeneous equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n y = 0 \qquad \ldots(1)$$

is equivalent to

$$(D-m_1)(D-m_2) \ldots\ldots\ldots (D-m_n)\, y = 0. \qquad \ldots(2)$$

The solution of any one of the equations

$$(D-m_1)\, y=0,\ (D-m_2)\, y=0,\ldots\ldots,\ (D-m_n)\, y=0 \qquad (3)$$

is also a solution of (2). For instance let $y=\phi_2\,(x)$ be a solution of the equation $(D-m_2)\ y=0$. Then on substituting $\phi_2\,(x)$ for y in the left-hand-side of (2), we get

$$\begin{aligned} f(D)\,\phi_2 &= (D-m_1)\,(D-m_3)\ldots\ldots(D-m_n)\,(D-m_2)\,\phi_2 \\ &= (D-m_1)\,(D-m_3)\ldots\ldots(D-m_n)\,(0) \\ &= 0, \end{aligned}$$

since the operational factors can be put down in any order we like.

Now the solution of $(D-m)\ y=0$ or of $\dfrac{dy}{dx}=my\ \left[i.e. \dfrac{dy}{y}=mdx\right]$

is given by $$y=c\,e^{mx},$$

where c is an arbitrary constant.

Hence the equations (3) have the solutions

$$y=c_1\,e^{m_1 x},\ y=c_2\,e^{m_2 x},\ldots\ldots\ldots,\ y=c_n\,e^{m_n x},$$

where $c_1, c_2,\ldots\ldots\ldots, c_n$ are arbitrary constants.

Therefore, the homogeneous equation (1) has the solution

$$y=c_1\,e^{m_1 x}+c_2\,e^{m_2 x}+\ldots\ldots+c_n\,e^{m_n x}. \qquad \ldots(4)$$

If the numbers $m_1, m_2,\ldots\ldots\ldots, m_n$ are *distinct*, the functions $e^{m_1 x}, e^{m_2 x},\ldots\ldots\ldots, e^{m_n x}$ are linearly independent and (4) is the general solution of (1).

In practice, the numbers $m_1, m_2,\ldots\ldots\ldots\ldots\ldots, m_n$ are found by solving for m the equation

$$m^n+p_1\,m^{n-1}+\ldots\ldots+p_n=0. \qquad \ldots(5)$$

Equation (5) is called the *auxiliary equation*, and is obtained by putting $D=m$ in $f\,(D)=0$.

Ex. 1. *Solve* : $2\,\dfrac{d^2y}{dx^2}+9\,\dfrac{dy}{dx}-18\,y=0.$

Here the auxiliary equation is

$$2m^2+9m-18=0\,;$$

solving for m we get, $m=-6, \frac{3}{2}$.

Hence the general solution of the differential equation is

$$y=c_1\,e^{-6x}+c_2\,e^{3x/2}.$$

EXERCISE III (A)

Solve the following equations :

1. $(D^2-n^2)\,y=0.$

2. $(D^3-2D^2-D+2)\,y=0.$

3. $2\dfrac{d^2x}{dt^2}+5\dfrac{dx}{dt}-12x=0.$

4. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-54\,y=0.$

5. $9\dfrac{d^2x}{dy^2}+18\dfrac{dx}{dy}-16\,x=0.$

3·4. Case of the Auxiliary Equation having Equal Roots.

In the above we have taken the roots $m_1, m_2, \ldots\ldots, m_n$ of the auxiliary equation (5) to be distinct. When two roots are equal, for instance m_1 and m_2, the solution (4) becomes

$$y=(c_1+c_2)\,e^{m_1 x}+c_3\,e^{m_3 x}+\ldots\ldots+c_n\,e^{m_n x}.$$

But, since c_1+c_2 can be replaced by a single constant c, this solution has only $n-1$ arbitrary constants ; and so is not the general solution.

The corresponding part of the solution is the solution of

$$(D-m_1)^2\,y=0.$$

On writing this in the form

$$(D-m_1)\,\{(D-m_1)\,y\}=0$$

and putting u for $(D-m_1)\,y$, the above becomes

$$(D-m_1)\,u=0,$$

the solution of which is

$$u=c_1\,e^{m_1 x}$$

Putting this value of u,

$$(D-m_1)\,y=c_1\,e^{m_1 x}.$$

This is a linear equation of the first order ; its integrating factor is $e^{-m_1 x}$ and the solution is

$$y\,e^{-m_1 x}=c_2+\int c_1\,e^{m_1 x}\,.\,e^{-m_1 x}\,dx$$

$$=c_2+c_1\,x.$$

$$\therefore\quad y=(c_2+c_1\,x)\,e^{m_1 x}.$$

If three roots of the auxiliary equation are equal, say $m_1=m_2=m_3$, by similar reasoning it can be shown that the corresponding solution is

$$y=e^{m_1 x}\,(A+Bx+Cx^2)+c_4\,e^{m_4 x}+\ldots\ldots+c_n\,e^{m_n x},$$

and if r roots are equal, the solution is

$$y=e^{m_1 x}\ (A_1+A_2 x+\ldots\ldots+A_r x^{r-1})+c_{r+1}\ e^{m_{r+1} x}$$
$$+\ldots\ldots+c_n\ e^{m_n x}.$$

Ex. 2. *Solve* : $\dfrac{d^4y}{dx^4}-\dfrac{d^3y}{dx^3}-9\ \dfrac{d^2y}{dx^2}-11\dfrac{dy}{dx}-4y=0.$

We have as the auxiliary equation

$$(m^4-m^3-9m^2-11m-4)\ y=0$$

or

$$(m+1)^3\ (m-4)\ y=0.$$

Hence the solution is given by

$$y=(A+Bx+Cx^2)\ e^{-x}+c_1\ e^{4x},$$

EXERCISE III (B)

Solve the following equations :

1. $(D^3+D^2-5D+3)\ y=0.$ 2. $\dfrac{d^2y}{dx^2}+2\ \dfrac{dy}{dx}+y=0.$

3. $(D^3-3D^2+3D-1)\ y=0.$ 4. $\dfrac{d^4y}{dx^4}-2\ \dfrac{d^2y}{dx^2}+y=0.$

3·5. Case of the auxiliary Equation having Complex Roots.

If the auxiliary equation has a pair of complex roots, say $m_1=\alpha+i\beta$, $m_2=\alpha-i\beta$, (where i stands for $\sqrt{-1}$, the corresponding part of the solution is

$$c_1\ e^{(\alpha+i\beta)\ x}+c_2\ e^{(\alpha-i\beta)\ x}$$

or

$$e^{\alpha x}\left(c_1\ e^{i\beta x}+c_2\ e^{-i\beta x}\right).$$

In order to put the above expression in a more convenient form we make use of the results

$$e^{i\theta}=\cos\theta+i\sin\theta,$$
$$e^{-i\theta}=\cos\theta-i\sin\theta.$$

The two terms of the solution can then be written

$$e^{\alpha x}\ [\ c_1\ (\cos\beta x+i\sin\beta x)+c_2\ (\cos\beta x-i\sin\beta x)\ ]$$
$$e^{\alpha x}\ [\ (c_1+c_2)\ \cos\beta x+i\ (c_1-c_2)\ \sin\beta x\ ]$$
$$e^{\alpha x}\ (A\cos\beta x+B\sin\beta x),$$

where $A=c_1+c_2$ and $B=i\ (c_1-c_2)$ are arbitrary constants.

If the pair of imaginary roots $\alpha+i\beta$ and $\alpha-i\beta$ occur twice, the corresponding part of the solution is

$$(c_1+c_2 x)\, e^{(\alpha+i\beta)x}+(c_3+c_4 x)\, e^{(\alpha-i\beta)x}$$

which reduces to

$$e^{\alpha x}\,[\,(A_1+A_2 x)\cos\beta x+(B_1+B_2 x)\sin\beta x\,]$$

Ex 3. *Solve :* $\dfrac{d^3y}{dx^3}-8y=0.$

The auxiliary equation is $m^3-8=0$, the roots of which are $m=2$ and $m=-1\pm i\sqrt{3}$.

Hence the solution is

$$y=e^{-x}\,(c_1\cos\sqrt{3}\,x+c_2\sin\sqrt{3}\,x)+c_3\,e^{2x}.$$

EXERCISE III (C)

Solve the following equations :

1. $(D^4+8D^2+16)\,y=0.$
2. $\dfrac{d^3y}{dx^3}+3\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-5y=0.$
3. $2\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+2y=0.$
4. $\dfrac{d^4y}{dx^4}-y=0.$
5. $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=0.$

3·6. Particular Integral. The Symbolic Operator $\dfrac{1}{f(D)}$.

We have already pointed out that the general solution of a linear differential equation with constant coefficients is the sum of the complementary Function [which is the solution of the equation obtained by substituting zero for Q] and the Particular Integral [which is a solution of the equation not containing any arbitrary constant]

Methods for finding the particular integral will now be sketched. It is necessary to extend the notation associated with the symbol D for discussing methods of computing it.

The expression $\dfrac{1}{f(D)}\,Q$ will be used to denote function of x, not involving arbitrary constants, which when operated upon by $f(D)$ gives Q. For example

$$\frac{1}{D^2-D}(2-2x)=x^2,$$

Since $(D^2-D)\,x^2=2-2x.$

In view of this definition, $\dfrac{1}{f(D)}$ may be regarded as representing the operation which is the inverse of that represented by $f(D)$.

Thus, in particular, when $f(D)$ stands for D, we get

$$\frac{1}{D}\,Q=\int Q\,dx$$

It is obvious that the function $\dfrac{1}{f(D)}Q$ is a particular integral of the differential equation

$$f(D)\,y=Q.$$

Now, $f(D)$ can be broken up into factors which may be taken in any order. Thus if

$$f(D)\equiv(D-m_1)\,(D-m_2)\ldots\ldots\ldots(D-m_n),$$

then we have

$$\frac{1}{f(D)}\,Q=\frac{1}{D-m_1}\cdot\frac{1}{D-m_2}\ldots\ldots\ldots\frac{1}{D-m_n}\,Q,$$

where the expression on the right is defined to mean that Q is first operated upon with $\dfrac{1}{D-m_n}$ then the result is operated upon with $\dfrac{1}{D-m_{n-1}}$ and so on until all the n operational factors have been utilised.

It is also possible to resolve $\dfrac{1}{f(D)}$ into the partial fractions

$$\frac{N_1}{D-m_1}+\frac{N_2}{D-m_2}+\ldots\ldots\ldots+\frac{N_n}{D-m_n},$$

so that

$$\frac{1}{f(D)}\,Q=\frac{N_1}{D-m}\,Q+\frac{N_2}{D-m_2}\,Q+\ldots\ldots+\frac{N_n}{D-m_n}\,Q.$$

Since both the methods consist of operations of the kind effected by $\frac{1}{D-a}$ upon Q, the result of this operation should be found and committed to memory.

$$\text{Let } u=\frac{1}{D-a}\, Q\ ;$$

then the result of operating upon u with $D-a$ is Q ;

i.e. $$(D-a)\, u=Q,$$

i.e. $$\frac{du}{dx}-a\, u=Q.$$

This is a linear equation in u, and its solution is

$$u\, e^{-ax} = C+\int e^{-ax}\, Q\, dx$$

or $$u=C\, e^{ax}+e^{ax}\int Q\, e^{-ax}\, dx.$$

Now it has already been pointed out that u is free from arbitrary constants, and hence

$$u=e^{ax}\int Q\ e^{-ax}\, dx.$$

We are now in a position to evaluate $\frac{1}{f(D)}\, Q$ in either of the two following ways :

(*i*) The operator $\frac{1}{f(D)}$ may be factorized ; then the particular integral

$$=\frac{1}{D-m_1}\cdot\frac{1}{D-m_2}\cdots\cdots\cdots\frac{1}{D-m_n}\, Q.$$

On operating with the first symbolic factor on the right, this

$$=\frac{1}{D-m_1}\cdot\frac{1}{D-m_2}\cdots\cdots e^{m_n x}\int e^{-m_n x}\, Q\, dx\ ;$$

then, on operating with the second and remaining factors in succession we finally get the particular integral

$$e^{m_1 x}\int e^{(m_2-m_1)x}\int\cdots\cdots\int e^{-m_n x}\, Q\, dx\ dx\cdots\cdots dx$$

(*ii*) The operator $\dfrac{1}{f(D)}$ may be separated into partial fractions

$$\frac{N_1}{D-m_1}+\frac{N_2}{D-m_2}+\dots\dots+\frac{N_n}{D-m_n},$$

and then the particular integral

$$\frac{1}{f(D)}\,Q=\frac{N_1}{D-m_1}\,Q+\dots\dots+\frac{N_n}{D-m_n}\,Q$$

$$=N_1\,e^{m_1 x}\int e^{-m_1 x}\,Q\,dx+\dots\dots\dots$$

$$+N_n\,e^{m_n x}\int e^{-m_n x}\,Q\,dx.$$

Ex. 4. *Solve* : $(D^3-2D^2+D)\,y=e^{-x}$.

Here the auxiliary equation is

$$m^3-2m^2+m=0$$

$\therefore\quad m=0,\ m=1$ repeated twice.

$\therefore$ The complementary function is

$$c_1+(c_2+c_3\,x)\ e^{x}.$$

The particular integral

$$=\frac{1}{D^3-2D^2+D}\,e^{-x}$$

$$=\left[\frac{1}{D}-\frac{1}{D-1}+\frac{1}{(D-1)^2}\right]e^{-x}$$

$$=\frac{1}{D}\,e^{-x}-\frac{1}{D-1}\,e^{-x}+\frac{1}{(D-1)(D-1)}\,e^{-x}$$

$$=\int e^{-x}\,dx-e^{x}\int e^{-x}\,e^{-x}\,dx+\frac{1}{D-1}\left[e^{x}\int e^{-x}e^{-x}\,dx\right]$$

$$=e^{-x}+\tfrac{1}{2}\,e^{-x}-\tfrac{1}{2}\frac{1}{D-1}\,e^{-x}$$

$$=-e^{-x}+\tfrac{1}{2}\,e^{-x}+\tfrac{1}{4}\,e^{-x},\ \text{since}\ \frac{1}{D-1}\,e^{-x}=-\tfrac{1}{2}\,e^{-x}.$$

$$=-\tfrac{1}{4}\,e^{-x}$$

$\therefore$ The complete solution is

$$y=c_1+(c_2+c_3\,x)\,e^{-x}-\tfrac{1}{4}\,e^{-x}.$$

Ex. 5. *Solve* : $\dfrac{d^2y}{dx^2}+n^2y=\sec nx.$ (*Nagpur* 1957)

The auxiliary equation is

$$m^2+n^2=0,\ \text{ or } m=\pm i\,n,$$

$\therefore$ The complementary function is

$$c_1\cos nx+c_2\sin nx.$$

The particular integral

$$=\frac{1}{D^2+n^2}\sec nx=\frac{1}{(D+in)(D-in)}\sec nx$$

$$=\frac{1}{2in}\left\{\frac{1}{D-in}-\frac{1}{D+in}\right\}\sec nx$$

Now $\dfrac{1}{D-in}\sec nx=e^{inx}\displaystyle\int\frac{e^{-inx}}{\cos nx}\,dx$

$$=e^{inx}\int\frac{\cos nx-i\sin nx}{\cos nx}\,dx$$

$$=e^{inx}\{\,x+i\,(1/n)\log\cos nx\,\}$$

Similarly $\dfrac{1}{D+in}\sec nx=e^{-nx}\{\,x-i\,(1/n)\log\cos nx\,\}$

$\therefore$ $P.\,I.=(1/n)\{\,x\sin nx+(1/n)\,(\log\cos nx)\cos nx\,\}$

$\therefore$ The complete solution is

$$y=c_1\cos nx+c_2\sin nx+\frac{x\sin nx}{n}+\frac{\cos nx\log\cos nx}{n^2}.$$

EXERCISE III (D)

Solve the following equations :

1. $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=e^{4x}$ 2. $\dfrac{d^2y}{dx^2}-y=2+5x$

3. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-15y=15x^2$ 4. $\dfrac{d^2y}{dx^2}+y=\sec^2x$ (*I.A.S.* 1953)

3·7. Short methods of computing the particular integral.

The general method which has been explained in the preceding article for finding the particular integral generally leads to laborious

calculation. This can often be avoided by use of the short methods to be given in this and the following article.

Particular Integral corresponding to the form e^{ax} of Q. The particular integral of the differential equation $f(D)\ y=e^{ax}$ is

$\dfrac{1}{f(D)}\ e^{ax}$, this will be shown to be equal to $\dfrac{1}{f(a)}\ e^{ax}$

We have

$$D\ e^{ax}=a\ e^{x},\ D^2\ e^{ax}=a^2\ e^{ax},\ D^{n-1}\ e^{ax}=a^{n-1}\ e^{ax},\ D^n\ e^{ax}=a^n\ e^{ax}.$$

$$\therefore\ f(D)\ e^{ax}\equiv(D^n+P_1\ D^{n-1}+\dots\dots\dots+P_{n-1}\ D+P_n)\ e^{ax}$$
$$=(a^n+P_1\ a^{n-1}+\dots\dots\dots+P_{n-1}\ a+P_n)\ e^{ax},$$

or $f(D)\ e^{ax}=f(a)\ e^{ax}$.

Let $f(a)\neq 0$.

Operating on both sides with $\dfrac{1}{f(D)}$,

$$\frac{1}{f(D)}\ \{f(D)\ e^{ax}\}=\frac{1}{f(D)}\ \{f(a)\ e^{ax}\}$$

Then, since $\dfrac{1}{f(D)}$ and $f(D)$ are inverse operators, and $f(a)$ is merely an algebraic multiplier which is $\neq 0$, the above reduces to

$$e^{ax}=f(a).\ \frac{1}{f(D)}\ e^{ax},$$

$$\therefore\ \frac{1}{f(D)}\ e^{ax}=\frac{1}{f(a)}\ e^{ax}.$$

The method fails if $f(a)=0$, in this case Art. 6·9 can be applied.

Ex. 6. *Solve* : $\dfrac{d^3y}{dx^3}+y=(e^x+1)^2$.

Written in symbolic form, this equation becomes

$$(D^3+1)\ y=e^{2x}+2e^x+1,$$

or $(D+1)\ (D^2-D+1)\ y=e^{2x}+2e^x+1$.

Here the roots of $f(m)=0$ are -1, $\frac{1}{2}\ (1\pm i\sqrt{3})$ hence the complementary function is

$$c_1\ e^{-x}+e^{x/2}\ (c_2\cos\tfrac{1}{2}\sqrt{3}\ x+c_3\sin\tfrac{1}{2}\sqrt{3}\ x)$$

The particular integral

$$=\frac{1}{D^3+1}\ (e^{2x}+2e^x+e^{0.x})$$

$$=\frac{1}{2^3+1}e^{2x}+2\cdot\frac{1}{1^3+1}e^x+\frac{1}{0+1}e^{0.x}$$

$$=\tfrac{1}{9}e^{2x}+e^x+1$$

Hence the complete solution is

$$y=c_1e^{-x}+e^{x/2}(c_2\cos\tfrac{1}{2}\sqrt{3}\,x+c_3\sin\tfrac{1}{2}\sqrt{3}\,x)+\tfrac{1}{9}e^{2x}+e^x+1.$$

EXERCISE III (E)

1. Find the particular integral of Q. 1, Exercise VI (D) by the short method.

Solve the following equations :

2. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=2e^{5x/2}$. 3. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=e^{-x}$.

4. $\dfrac{d^2y}{dx^2}+2p\dfrac{dy}{dx}+(p^2+q^2)\,y=e^{kx}$.

3·1. Short method of finding $\{(1/f(D))\}$ sin ax and $\{1/f(D)\}$ cos ax, when $\phi(-a^2)\neq 0$. Differentiation of sin ax gives

$$D\sin ax=a\cos ax,$$
$$D^2\sin ax=-a^2\sin ax;$$
$$D^3\sin ax=-a^3\cos ax,$$
$$D^4\sin ax=+a^4\sin ax,$$

and, in general,

$$(D^2)^n\sin ax=(-a^2)^n\sin ax.$$

Hence if $f(D)$ contains only even powers of D and we denote it by $\phi(D^2)$ it is clear that

$$\phi(D^2)\sin ax=\phi(-a^2)\sin ax,$$

Operating on both sides with $\dfrac{1}{\phi(D^2)}$

$$\sin ax=\frac{1}{\phi(D^2)}\{\phi(-a^2)\sin ax\}$$

or, since $\phi(-a^2)$ is an algebraic multiplier,

$$\frac{1}{\phi(D^2)}\sin ax=\frac{1}{\phi(-a^2)}\sin ax.$$

Similarly, it can be shown that

$$\frac{1}{\phi(D^2)}\cos ax=\frac{1}{\phi(-a^3)}\cos ax\ ;$$

and, more generally, that

$$\frac{1}{\phi(D^2)}\sin(ax+b)=\frac{1}{\phi(-a^2)}\sin(ax+b).$$

When $f(D)$ contains also odd powers of D, the method to be followed is as shown in the following example.

Ex. 7. *Solve* : $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y=a\cos 2x.$

Here $f(D)=D^2-4D+1$

$\therefore$ The roots of $f(m)=0$ are $2\pm\sqrt{3}$.

Hence the complementary function is

$$e^{2x}\left(c_1 e^{\sqrt{3}\,x}+c_2 e^{-\sqrt{3}\,x}\right)$$

The particular integral

$$=\frac{1}{D^2-4D+1}a\cos 2x=a\cdot\frac{1}{(-2^2)-4D+1}\cos 2x$$

$$=-a\cdot\frac{1}{4D+3}\cos 2x=-a\cdot\frac{(4D-3)}{(4D+3)(4D-3)}\cos 2x,$$

since $(4D-3)$ and $\dfrac{1}{4D-3}$ are inverse operators ;

$$=-a\cdot\frac{(4D-3)}{16D^2-9}\cos 2x=-a\cdot\frac{4D-3}{16(-2^2)-9}\cos 2x$$

$=(a/73)(4D-3)\cos 2x=(a/73)\cdot(-4\cdot 2\cdot\sin 2x-3\cos 2x)$
$=-(a/73)(8\sin 2x+3\cos 2x).$

$\therefore$ The complete solution is

$$y=e^{2x}\left(c_1 e^{\sqrt{3}\,x}+c_2 e^{-\sqrt{3}\,x}\right)-(a/73)(8\sin 2x+3\cos 2x).$$

3·9. $\{1/f(D)\}\sin ax$, **exceptional case.** The method indicated in the preceding article fails if $f(-a^2)=0$; this will be so when D^2+a^3 is a factor of $f(D)$. We have then to apply the general method of Art. 6·6.

Let $f(D)=(D^2+a^2)\cdot\psi(D)$, then to evaluate $\dfrac{1}{f(D)}\sin ax$ we can first calculate $\dfrac{1}{D^2+a^2}\sin ax$ and then operate the result with $\dfrac{1}{\psi(D)}$.

Now $\dfrac{1}{D^2+a^2}\sin ax=\dfrac{1}{(D-ia)(D+ia)}\sin ax$

$$=\frac{1}{2ia}\left\{\frac{1}{D-ia}-\frac{1}{D+ia}\right\}\sin ax.$$

But $\dfrac{1}{D-ia}\sin ax=e^{iax}\displaystyle\int e^{-iax}\sin ax\,dx$

$$=e^{iax}\int e^{-iax}\frac{(e^{iax}-e^{-iax})}{2i}\,dx$$

$$=\frac{e^{iax}}{2i}\int(1-e^{-2iax})\,dx=\frac{e^{iax}}{2i}\left(x+\frac{e^{-2iax}}{2ia}\right)$$

Similarly $\dfrac{1}{D+ia}\sin ax=-\dfrac{e^{-iax}}{2i}\cdot\left\{x-\dfrac{e^{2iax}}{2ia}\right\}$

$$\therefore\quad\frac{1}{D^2+a^2}\sin ax=\frac{1}{2ia}\left\{\frac{\cos ax}{i}\,x-\frac{\sin ax}{2ia}\right\}$$

$$=-\frac{x\cos ax}{2a}+\frac{\sin ax}{4a^2}.$$

Now, since (D^2+a^2) is a factor of $f(D)$, the complementary function contains the terms $c_1\sin ax+c_2\cos ax$ and hence the term $\dfrac{\sin ax}{4a^2}$ in the particular integral found above can be absorbed in the term $c_1\sin ax$ of the complementary function. Hence we may write

$$\frac{1}{D^2+a^2}\sin ax=-\frac{x}{2a}\cos ax.$$

Similarly it can be proved that

$$\frac{1}{D^2+a^2}\cos ax=\frac{x}{2a}\sin ax.$$

Ex. 8. *Solve* : $\dfrac{d^3y}{dx^3}+a^2\dfrac{dy}{dx}=\sin ax.$

The auxiliary equation is $m^3+a^2m=0$,

$\therefore$ The roots of m are 0 and $\pm ia$.

Hence the complementary function is $c_1e^{0x}+c_2\sin ax+c_3\cos ax$ *i.e.* $c_1+c_2\sin ax+c_3\cos ax$.

The particular integral$=\dfrac{1}{D}\cdot\dfrac{1}{D^2+a^2}\sin ax$

$$=\frac{1}{D}\left(-\frac{x}{2a}\cos ax\right)=-\frac{x\sin ax}{2a^2}-\frac{\cos ax}{2a^3}$$

Hence the complete solution is

$$y=c_1+c_2\sin ax+c_3\cos ax-(x\sin ax)/2a^2$$

(The second term in the particular integral has been omitted as it can be absorbed in the term $c_3\cos ax$ of the complementary function).

Ex. 9. *Solve* : $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-2y=e^x+\cos x$

(*Agra* 1960)

The auxiliary equation is

$$m^3-3m^2+4m-2=0 \quad \text{or} \quad m=1,\ 1\pm i$$

Hence the *C. F.* is $c_1e^x+c_2e^x\cos(x+\alpha)$.

The $P.I.=\dfrac{1}{(D-1)(D^2-2D+2)}(e^x+\cos x)$

$$=\frac{1}{(D-1)(1-2+2)}e^x+\frac{1}{(-1-2D+2)(D-1)}\cos x$$

$$=e^x.\ \frac{1}{(D+1-1)}\cdot 1+\frac{1}{-2D^2+3D-1}\cos x$$

$$=x\ e^x+\frac{1}{3D+1}\cos x=x\ e^x+\frac{3D-1}{9D^2-1}\cos x$$

$$=x\ e^x+\frac{3D-1}{-9-1}\cos x=x\ e^x-\frac{1}{10}(3D-1)\cos x$$

$$=x\ e^x+\frac{1}{10}(3\sin x+\cos x).$$

Hence the complete solution is

$$y=c_1e^x+c_2e^x\cos(x+\alpha)+x\ e^x+\frac{1}{10}(3\sin x+\cos x).$$

EXERCISE III (F)

Solve the following equations :

1. $\dfrac{d^2y}{dx^2}+9y=\cos 2x+\sin 2x$

2. $(D^2+a^2)\,y=\cos ax+\cos bx$ (*Nagpur* 1958)

3. $\dfrac{d^2y}{dx^2}+4y=e^x+\sin 2x$ 4. $(D^3+3D^2-4D-12)\,y=\cos 4x$

5. $\dfrac{d^2y}{dx^2}-4y=2\sin \frac{1}{2}x$ 6 $\dfrac{d^3y}{dx^3}+y=\sin 3x-\cos^2 \frac{1}{2}x.$

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3 10. Evaluation of $\dfrac{1}{f(D)}\,xV$, V being any function of x

We have $D\,(xV)=x.\,DV+V,$

$$D^2\,(xV)=x.\,D^2V+2DV=x.\,D^2V+\left(\frac{d}{dD}\,D^2\right)V,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$D^n(xV)=xD^n\,V+nD^{n-1}\,V, \text{ by Leibnitz's theorem,}$$

$$=xD^nV+\left(\frac{d}{dD}\,D^n\right)V.$$

Therefore $f\,(D)\,xV=x.\,f\,(D)\,V+f'\,(D)\,V.$

Now, put $f\,(D).\,V=V_1$; then $V=\dfrac{1}{f(D)}\,V_1.$

Substituting this value of V in (1) we obtain

$$f(D)\,x\,\frac{1}{f(D)}\,V_1=x\,V_1+f'(D)\,\frac{1}{f(D)}\,V_1$$

Operating on this equation with $\dfrac{1}{f(D)}$ we get

$$x.\frac{1}{f(D)}\,V_1=\frac{1}{f(D)}\,x\,V_1+\frac{1}{f(D)}\,f'\,(D)\,\frac{1}{f(D)}\,V_1$$

$$=\frac{1}{f(D)}\,xV_1+f'\,(D)\,\frac{1}{\{f(D)\}^2}\,V_1.$$

Transposing

$$\frac{1}{f(D)}\,x\,V_1=x.\,\frac{1}{f(D)}\,V_1-f'\,(D)\,\frac{1}{\{f(D)\}^2}\,V_1$$

or, on dropping the suffix of V_1,

$$\frac{1}{f(D)}\,x\,V=x.\,\frac{1}{f(D)}\,V+\left\{\frac{d}{dD}\left(\frac{1}{f(D)}\right)\right\}V$$

The evalution of $\frac{1}{f(D)}\ x^m V$, where m is a positive integer, can be done by repeated application of the above method.

$$\text{Thus}\ \frac{1}{f(D)}\ x^m\ V = x^m . \frac{1}{f(D)}\ V + mx^{m-1} . \left\{ \frac{d}{dD}\ \frac{1}{f(D)} \right\} V$$

$$+ \frac{m\,(m-1)}{2\,!}\ x^{m-2} . \left\{ \frac{d^2}{dD^2}\ \frac{1}{f\,(D)} \right\}\ V + \ldots\ldots$$

Note :—The foregoing formulae are not always convenient in practice, The student is advised to try the earlier methods rather than apply the above results.

We give below an illustration, an alternative solution of which will be given in Art. 3·13.

Ex. 10. *Solve* : $\frac{d^2y}{dx^2} - 2\ \frac{dy}{dx} + y = x \sin x.$

The complementary function is $(c_1 + c_2\ x)\ e^x$.

The particular integral

$$= \frac{1}{D^2 - 2D + 1}\ \ x \sin x$$

$$= x . \frac{1}{D^2 - 2D + 1}\ \sin x - 2\ (D-1) . \frac{1}{(D^2 - 2D + 1)^2}\ \sin x$$

$$= x . \left(-\frac{1}{2D}\ \sin x \right) - 2\ (D-1)\ \frac{1}{4D^2}\ \ \sin x.$$

$$= \tfrac{1}{2}\ x \cos x + \tfrac{1}{2}\ (D-1) \sin x = \tfrac{1}{2}\ (x \cos x + \cos x - \sin x).$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2\ x)\ e^x + \tfrac{1}{2}\ (x \cos x + \cos x - \sin x).$$

3·11. Short method of finding $\{ 1/f\ (D) \}\ x^m$, m a positive integer.

In order to evaluate $\frac{1}{f\ (D)}\ x^m$, expand $\frac{1}{f\ (D)}$ in ascending powers of D and operate on x^m with the result. It is clear that terms of the expansion beyond the m^{th} power of D need not be written since $D^{m+1}\ x^m = 0$.

Ex. 11. *Solve* : $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2.$

The auxiliary equation is $m^3 + 3m^2 + 2m = 0$; its roots are $m = 0, -1, -2$.

Hence the complementary function is $c_1 + c_3\ e^{-x} + c_3\ e^{-2x}$

The particular integral

$$=\frac{1}{D^3+3D^2+2D}x^2=\frac{1}{2D}\cdot\frac{1}{\left(1+\frac{D^2+3D}{2}\right)}x^2$$

$$=\frac{1}{2D}\left(1+\frac{D^2+3D}{2}\right)^{-1}x^2$$

$$=\frac{1}{2D}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2-\ldots\ldots\right]x^2$$

$$=\frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\tfrac{1}{4}(D^4+6D^3+9D^2)+\ldots\ldots\right]x^2$$

$$=\frac{1}{2D}\left(1-\frac{3D}{2}+\frac{7}{4}D^2+\ldots\ldots\right)x^2$$

$$=\frac{1}{2}\left(\frac{1}{D}-\frac{3}{2}+\frac{7}{4}D\right)x^2$$

$$=\frac{1}{2}\left[\frac{1}{3}x^3-\frac{3}{2}x^2+\frac{7}{4}\cdot 2x\right]$$

$$=\frac{1}{12}(2x^3-9x^2+21x).$$

$\therefore$ The complete solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x}{12}(2x^2-9x+21)$$

EXERCISE III (G)

Solve the following equations :

1. $\dfrac{d^3y}{dx^3}-4\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-2y=0$
2. $y'''-y''-y'+y=x$
3. $(D^3-D^2-6D)\,y=1+x^2$
4. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^{2x}+x^2+x$

1.23. Short method of computing $\{1/f(D)\}\,e^{ax}V$, V being any function of x. We notice that

$$D(e^{ax}V)=e^{ax}.DV+ae^{ax}.V=e^{ax}(D+a)V.$$

and $D^2(e^{ax}V)=ae^{ax}(D+a)V+e^{ax}D(D+a)V=e^{ax}(D+a)^2 \cdot V$;
and, in general, as is seen from successive differentiation,

$$D^n e^{ax} V=e^{ax}(D+a)^n V;$$

therefore, $f(D)\,e^{ax}V=e^{ax}f(D+a)\,V$ (1)

This suggests that we may write

$$\frac{1}{f(D)}\,e^{ax}V=e^{ax}\cdot\frac{1}{f(D+a)}V.$$

We can easily verify that the proposition is really true. For, operating on the right-hand-side with $f(D)$ we have

$$f(D)\left[\,e^{ax}\left\{\frac{1}{f(D+a)}V\right\}\right]$$

$=f(D)\,[\,e^{ax}\cdot V_1\,]$, where V_1 stands for the function $\dfrac{1}{f(D+a)}V$;

$=e^{ax}\cdot f(D+a)\cdot V_1$, by (1);

$=e^{ax}\cdot f(D+a)\left\{\dfrac{1}{f(D+a)}V\right\}$, on restoring the value of V_1

$=e^{ax}\cdot V$, since $f(D+a)$ and $\dfrac{1}{f(D+a)}$ are inverse operators.

This proposition enables us to find $\dfrac{1}{f(D)}e^{ax}$ when $f(a)=0$, as in the solved Ex. 12 below.

Ex. 12. *Solve*: $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x^2e^{3x}$.

The auxiliary equation is $m^2-2m+1=0$. Its roots are $m=1$, repeated twice.

$\therefore$ The complementary function is $(c_1+c_2x)\,e^x$

The particular integral

$$=\frac{1}{D^2-2D+1}x^2e^{3x}$$

$=e^{3x}\cdot\dfrac{1}{(D+3)^2-2(D+3)+1}x^2$, by the formula of this article;

$$=e^{3x}\cdot\frac{1}{D^2+4D+4}x^2=\frac{e^{3x}}{4}\cdot\left(1+\frac{D}{2}\right)^{-2}x^2$$

$$=\frac{e^{3x}}{4}\left(1-D+3.\quad\frac{D^2}{4}-\ldots\ldots\right)x^2$$

$$=\frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{4}\,.\,2\right)=\frac{e^{3x}}{8}\,(2x^2-4x+3)$$

$\therefore$ The complete solution is

$$y=(c_1+c_2\,x)\,e^x+\tfrac{1}{8}e^{3x}\,(2x^2-4x+3).$$

Ex. 13. *Solve :* $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=2\,\sinh 2x.$

The equation, written in the symbolic form, is

$$(D^2+4D+4)\,y=e^{2x}-e^{-2x}.$$

$\therefore$ The complementary function is $(c_1+c_2\,x)\,e^{-2x}$.

The $P.I.=\dfrac{1}{D^2+4D+4}\,(e^{2x}-e^{-2x})$

$$=\frac{1}{D^2+4D+4}\,e^{2x}-\frac{1}{D^2+4D+4}\,e^{-2x}.\,1$$

$$=\frac{1}{2^2+4\cdot2+4}\,e^{2x}-e^{-2x}\frac{1}{[(D-2)^2+4(D-2)+4]}\,.\,1$$

$$=\frac{1}{16}\,e^{2x}-e^{2x}\left(\frac{1}{D^2}\right)1=\frac{e^{2x}}{16}-e^{-2x}\,\frac{1}{D}\int 1\,dx$$

$$=\tfrac{1}{16}\,e^{2x}-e^{-2x}\int x\,dx=\tfrac{1}{16}\,e^{2x}-\tfrac{1}{2}\,x^2\,e^{-2x}$$

$\therefore$ The complete solution is

$$y=(c_1+c_2\,x)\,e^{-2x}+\tfrac{1}{16}\,e^{2x}-\tfrac{1}{2}\,x^2\,e^{-2x}$$

EXERCISE III (H)

Solve the following equations :

1. $(D+2)\,(D-1)^2\,y=e^x.$
2. $(D^2-2D+5)\,y=e^{2x}\sin x$
3. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+4y=e^x\cos x$
4. $(D^2-1)\,y=\cosh x\cos x$
5. $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dx}-6y=e^{2x}\,(1+x)$
6. $\dfrac{d^4y}{dx^4}+\dfrac{d^2y}{dx^2}+y=ax^2+be^{-x}\sin 2x$
7. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-12y=(x-1)\,e^{2x}$

3·13. Synopsis. Looking back on this chapter we find that the first step to solve a linear equation with constant coeffieients is to determine the complementary function. This can be done either by solving the auxiliary equation or by solving $f(D)=0$, treating D as an algebraical symbol. The student must have noticed that if the given differential equation is $f(D)\,y=Q$, the auxiliary equation is found to be $f(m)=0$, so that the roots of $f(m)=0$ in m are the same as the roots of $f(D)=0$ in D. If some of the roots are equal, the method of Art. 3·6 is to be followed. If a pair of roots is imaginary, the corresponding part of the complementary function can be put in an elegant form by employing trigonometrical functions. [Art. 3·5.]

The next step is to compute the particular integral. This can be done easily if the right hand member of the differential equation viz. Q, comes under any one of the following forms :

(*i*) e^{ax}, where a is any constant ; [Art. 3·7] ;

(*ii*) $\sin ax$, $\cos ax$, [Art. 3·8];

(*iii*) x^m, where m is a positive integer, [Art. 3·11] ;

(*iv*) $e^{ax}\,V$, where V is any function of x, [Art. 3·12].

If Q cannot be put under any one of these forms or in the case of failure of short methods, the general method of Art. 3·6 is to be applied. Of the two modes of procedure given there, the latter one is generally to be preferred.

In evaluating the particular integral corresponding to Q equal to $\cos ax$, or to $\sin ax$, or to expression in which $\sin ax$ or $\cos ax$ occurs as a factor (say $Q=x^m \sin ax$) it is sometimes more convenient to replace $\sin ax$ or $\cos az$ by the exponential value and apply Art. 3·7 or 3·12 proceeding as in the following worked out examples.

Ex. 14. *Solve* : $\dfrac{d^2y}{dx^2}+a^2y=\cos ax.$

The complementary function is $c_1 \sin ax+c_2 \cos ax$

The $P.I.=\dfrac{1}{D^2+a^2}\cos ax$

$$=\text{Real part of } \frac{1}{D^2+a^2}(\cos ax+i\sin ax)$$

$$=\text{Real part of } \frac{1}{D^2+a^2}e^{iax}$$

But $\dfrac{1}{D^2+a^2}\,e^{iax}=\dfrac{1}{D-ia}\left(\dfrac{1}{D+ia}\,e^{iax}\right)$

$$=\frac{1}{D-ia}\left(\frac{e^{iax}}{2ia}\right), \qquad \text{(by Art. 3·7)}$$

$$=\frac{1}{2ia}\cdot e^{iax}, \frac{1}{(D+ia)-ia}\cdot 1$$

$$=\frac{1}{2ia}\,e^{iax}.\,x=-\frac{ix}{2a}\,(\cos ax+i\sin ax)$$

$\therefore$ The real part $=\frac{x}{2a}\sin ax$.

$\therefore$ The complete solution is

$$y=c_1\sin ax+c_2\cos ax+\frac{x}{2a}\sin ax$$

[On equating the imaginary part we could get

$$\frac{1}{D^2+a^2}\sin ax=-\frac{x}{2a}\cos ax \;]$$

Ex. 14. *Solve* : $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x\sin x$.

The complementary function is easily found out as

$$(c_1+c_2\,x)\,e^x.$$

The particular integral

$$=\frac{1}{D^2-2D+1}\,x\sin x$$

$$=\text{The imaginary part of }\frac{1}{D^2-2D+1}\,x\,(\cos x+i\sin x)$$

$$=\text{The imaginary part of }\frac{1}{(D-1)^2}\,x\,e^{ix}$$

$$=e^{ix}\,\frac{1}{(D+i-1)^2}\,x$$

$$=\frac{e^{ix}}{k^2}\left\{\left(1+\frac{D}{x}\right)^{-2}x\right\},\text{ on putting } i-1=k$$

$$=\frac{e^{ix}}{k^2}\left\{\left(1-\frac{2D}{k}+\dots\dots\dots\right)x\right\}$$

$$=\frac{e^{ix}}{k^2}\left(x-\frac{2}{k}\right).$$

$$=\left(\frac{\cos x+i\sin x}{-2i}\right)\left[x+\frac{2}{2i}(i-1)\right],\ \therefore\ k^2=-2i$$

$$=\left(\frac{i\cos x}{2}-\frac{\sin x}{2}\right)(x+1+i)$$

$\therefore$ The imaginary part
$=\frac{1}{2}(x\cos x+\cos x-\sin x)$.

$\therefore$ The complete solution is
$y=(c_1+c_2 x)e^x+\frac{1}{2}(x\cos x+\cos x-\sin x)$.

Miscellaneous Examples on Chapter III

Solve :

1. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=x\cos x$.

2. $(D^4+2D^2+1)y=x^2\cos x$. (*I.A.S.* 1959)

3. $\dfrac{d^4y}{dx^4}-y=x\sin x$.

4. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=x e^x\sin x$

5. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=8x^2 e^{2x}\sin 2x$.

6. $\dfrac{d^2y}{dx^2}+y=e^{-x}+\cos x+x^3+e^x\sin x$.

7. $(D^4+D^2+1)y=e^{-x/2}\cos\left(x\dfrac{\sqrt{3}}{2}\right)$.

8. $(D-1)^2(D^2+1)^2 y=\sin^2\frac{1}{2}x+e^x$. (*Banaras* 1960)

9. $\dfrac{d^4y}{dx^4}+\dfrac{d^2y}{dx^2}+16y=16x^2+256$.

10. $(D^2+1)y=3\cos^2 x+2\sin^3 x$.

11. $(D^4+10D^2+9)y=96\sin 2x\cos x$.

12. $(D^5-13D^3+26D^2+82D+104)y=0$.

13. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+10y+37\sin 3x=0$, and find the value of y when $x=\frac{1}{2}\pi$ if it is given that $y=3$ and $\dfrac{dy}{dx}=0$ when $x=0$.

14. $(D^2+1)^2 y=24x\cos x$, given the initial conditions
$x=0,\ y=0,\ Dy=0,\ D^2y=0,\ D^3y=12$.

15. Show that $\dfrac{ND+M}{(D-a)^2+\beta^2}X$, where N and M are constants, is equal to twice the real part of

$\dfrac{1}{2i\beta}\left(\dfrac{1}{D-a-\beta i}\right)X$ operated on by $ND+M$; that is to

$$\frac{e^{ax}\sin\beta x}{\beta}\int e^{ax}\cos\beta x\ X\,dx-\frac{e^{ax}\cos\beta x}{\beta}\int e^{ax}\sin\beta x\ X\,dx.$$

CHAPTER IV

EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE

4·1. Equations Solvable for p. In this chapter we shall deal with those differential equations which involve dy/dx in a higher degree than one. For brevity we shall be using here p to denote dy/dx. Let us first consider an equation of the first order and n^{th} degree of the form

$$p^n+P_1\, p^{n-1}+P_2\, p^{n-2}+\ldots\ldots+P_{n-1}\, p+P_n=0$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y. Suppose that it can be solved for p and is of the form

$$[\, p-f_1\,(x, y)\,]\,[\, p-f_2\,(x, y)\,]\ldots\ldots[\, p-f_n\,(x, y)\,]=0.$$

Then each factor can be equated to zero and the solutions of the n equations thus formed can be obtained. Let the solutions be

$$F_1\,(x, y, c_1)=0,\ldots\ldots\ldots, F_n\,(x, y, c_n)=0 \qquad \ldots(1)$$

Combining these into one equation, the general solution is

$$F_1\,(x, y, c_1)\, F_2\,(x, y, c_2)\ldots\ldots F_n\,(x, y, c_n)=0 \qquad \ldots(2)$$

There is no loss of generality if the arbitrary constants $c_1, c_2, \ldots\ldots, c_n$ are replaced by a single arbitrary constant c because every particular solution obtainable from the equation (1) can also be obtained from (2) (if c_1, c_2, etc. are all replaced by c) by giving a suitable value to c.

Thus the general solution is

$$F_1\,(x, y, c)\, F_2\,(x, y, c)\ldots\ldots F_n\,(x, y, c)=0.$$

Ex. 1. *Solve* :

$$p^3-(x^2+xy+y^2)\, p^2+(x^3y+x^2y^2+xy^3)\, p-x^3y^3=0.$$

On factorization the given equation becomes

$$(p-y^2)\,(p-x^2)\,(p-xy)=0.$$

The first factor $p-y^2=0$ gives $xy+yc+1=0$ as its solution.
The second factor $p-x^2=0$ gives $x^3-3y+c=0$ as its solution.

The third factor $p-xy=0$ gives $e^{\frac{1}{2}x^2}+cy=0$ as its solution. Hence the complete primitive is

$$(xy+cy+1)\,(x^3-3y+c)\left(cy+e^{\frac{1}{2}x^2}\right)=0.$$

Ex. 2. *Solve* : $x^2(p^2-y^2)+y^2=x^4+2xyp$.

The equation is quadratic in p. Hence solving for p, we get

$$p=y/x \pm (x^2+y^2)^{1/2}.$$

Now substituting $y=vx$ and therefore $p=\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes $\frac{dv}{\sqrt{(1+v^2)}}=\pm dx$.

With the negative sign the solution is

$$v=\sinh(c-x) \text{ i. e. } y=x\sinh(c-x).$$

With the positive sign the solution is

$$v=\sinh(x+c), \text{ i.e. } y=x\sinh(x+c).$$

Hence the general solution is

$$[\,y-x\sinh(c-x)\,]\,[\,y-x\sinh(x+c)\,]=0.$$

Ex. 3. *Solve* : $(a^2-x^2)p^3+bx(a^2-x^2)p^2-p-bx=0$.

On factorization the equation becomes

$$(p+bx)(p\sqrt{a^2-x^2}-1)(x\sqrt{a^2-x^2}+1)=0$$

The first factor when equated to zero gives

$$y+\tfrac{1}{2}bx^2-c=0. \qquad \ldots\ldots(1)$$

The second factor when equated to zero gives

$$\frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}} \text{ whence } y=\sin^{-1}\frac{x}{a}+c$$

i.e.

$$x/a-\sin(y-c)=0 \qquad \ldots\ldots(2)$$

Similarly third factor when equated to zero gives

$$x/a+\sin(y-c)=0 \qquad \ldots\ldots(3)$$

Hence the general solution is

$$(y+\tfrac{1}{2}bx^2-c)\{x^2/a^2-\sin^2(y-c)\}=0.$$

Ex. 4. *Solve* : $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$.

On factorization the equation becomes

$$p(p+1)(px+2py+2x+y)=0$$

The first two factors when equated to zero give

$$y-c=0,\ y+x-c=0.$$

The third factor gives $p(x+2y)+2x+y=0$

i.e.

$$dy(x+2y)+(2x+y)dx=0$$

which is exact and its integration gives

$$x^2+xy+y^2-c=0.$$

Hence the general solution is

$$(y-c)(y+x-c)(x^2+xy+y^2-c)=0.$$

Ex. 5. *Solve :* $y-(1+p^2)^{-1/2}=b$.

solving for p, $p=\dfrac{\sqrt{1-(y-b)^2}}{y-b}$

whence $\dfrac{dy\,(y-b)}{\sqrt{1-(y-b)^2}}=dx$

Integrating, $\sqrt{1-(y-b)^2}=x+c$

i.e. $(x+c)^2+(y-b)^2=1$.

EXERCISE IV (A)

Solve :

1. $p^2-7p+12=0$.
2. $p^2-5p+6=0$ (*Dehli* 1959).
3. $p^2-9p+18=0$.
4. $p^2+2xp-3x^2=0$.
(*Luck.* 1938, *Raj* 1958 *Ban.* 1959).
5. $p^2+2py\cot x=y^2$.
6. $p^2-2p\cosh x+1=0$.
7. $p\,(p-y)=x\,(x+y)$.
(*Alld.* 1951, *Delhi Hon's.* 1954)
8. $yp^2+(x-y)\,p-x=0$.
9. $x+yp^2=p\,(1+xy)$.
(*Allahabad* 1932)
10. $xp^2+(y-x)\,p-y=0$.
11. $p^3-ax^4=0$.
12. $(p^2+px+py+xy)=0$.
13. $p^3-p\,(x^2+xy+y^2)+xy\,(x+y)=0$.
14. $(p+y+x)(xp+y+x)\,(p+2x)=0$.
15. $x^2p^3+y\,(1+x^2y)\,p^2+y^2\,p=0$.
(*Banaras* 1951)
16. $x^2p^2+xyp-6y^2=0$.
17. $p^3+2x\,p^2-y^2\,p^2-2xy^2\,p=0$.
(*Lucknow* 1949, 1951)
18. $p^2\,(2-3y)^2=4\,(1-y)$

4·21. Equations solvable for y. If the differential equation is solvable for y, it may be put in the form

$$y=f\,(x,\,p), \qquad \ldots\ldots(1)$$

Differentiation with respect to x gives an equation of the form

$$p=\phi\left(x,\,p,\,\frac{dp}{dx}\right)$$

which is an equation in two variables x and p. It may be possible to obtain its solution, say,

$$F\,(x,\,p,\,c)=0 \qquad \ldots\ldots(2)$$

The elimination of p between (1) and (2) will give the required solution, a relation involving x, y and c.

If the elimination of p between these equations is not feasible, the values of x and y may be obtained in terms of the parameter p. Let us put them in the form

$$x=F_1\,(p,\,c),\; y=F_2\,(p,\,c).$$

Then these two relations together constitute the solution.

4·22. Equations containing no x. Let the equation be of the form

$$f(y, p)=0.$$

(*i*) *If it is solvable for p*, let us have

$$p=\frac{dy}{dx}=F(y)$$

and its solution is $\int \frac{dy}{F(y)}=x+c.$

(*ii*) *If it is solvable for y*, let us have

$$y=f(p)$$

which can be integrated by the method of Art. 4·21.

Ex. 6. *Solve* : $y=x/p-ap$.

The equation is solved for y. Hence differentiating

$$p=\frac{1}{p}-\frac{x}{p^2}\frac{dp}{dx}-\frac{adp}{dx}$$

whence $$(ap^2+x)\frac{dp}{dx}=p(1-p^2)$$

or $$\frac{dx}{dp}-\frac{1}{p(1-p^2)}x=\frac{ap}{1-p^2}$$

The equation is linear which on integration gives

$$x=\frac{p}{\sqrt{1-p^2}}(a \sin^{-1} p+c)\ x \qquad \ldots\ldots(1)$$

Substituting this value of x in the original equation

$$y=-ap+\frac{1}{\sqrt{1-p^2}}(a \sin^{-1} p+c) \qquad \ldots\ldots(2)$$

Equations (1) and (2) constitute the general solution.

Ex. 7. *Solve* : $p^3+mp^2=a(y+mx)$.

Solving for y, $ay=p^3+mp^2-amx$.

On differentiation we get

$$ap=(3p^2+2mp)\frac{dp}{dx}-am,$$

whence $a\,dx=\frac{3p^2+2mp}{p+m}dp.$

On integration $ax+c=\frac{3}{2}p^2-mp+m^2 \log(p+m)$, which with the given relation is the general solution.

EXERCISE IV (B)

1. $y=3x+a \log p$.
2. $p^2-py+x=0$.
3. $y=x+a \tan^{-1} p$.
4. $3p^5-py+1=0$.
5. $y=p^2x+p$.
6. $xp^2+ax=2yp$.
7. $p^3+p=e^y$.
8. $y=\sin p-p \cos p$.
9. $y=p \sin x+\cos x$.
10. $y=p \tan p+\log \cos p$.

4·31. Equations solvable for x. If the given differential equation is solvable for x, let it be put in the form

$$x=f(y, p). \qquad \ldots\ldots(1)$$

Differentiation with respect to y gives an equation of the form

$$\frac{1}{p}=\phi\left(y, p, \frac{dp}{dy}\right)$$

which is an equation in two variables y and p. It may be possible to obtain its solution, say

$$F(y, p, c)=0 \qquad \ldots\ldots(2)$$

The elimination of p between (1) and (2) will give the required solution, a relation involving x, y and c, or as in the last Article x and y may be expressed in terms of the parameter p.

4·32. Equations containing no y. Let the equation be of the form

$$f(x, p)=0.$$

(*i*) *If it is solvable for* p, let us have

$$p=\frac{dy}{dx}=F(x)$$

and its solution is $y=\int F(x)\, dx+c$.

(*ii*) *If it is solvable for* x, let us have

$$x=F(p)$$

which can be solved by the method given before.

Ex. 8. *Solve* : $x p^3=a+bp$

Here $$x=a/p^3+b/p^2 \qquad \ldots\ldots(1)$$

Differentiating with respect to y

$$p^{-1}=(-3ap^{-4}-2bp^{-3})\frac{dp}{dy}$$

whence $$dy=\left(-\frac{3a}{p^3}-\frac{b}{p^2}\right)dp.$$

Therefore $$y=\frac{3a}{2p^2}+\frac{2b}{p}+c \qquad \text{......(2)}$$

Equations (1) and (2) constitute the general solution.

Ex. 9. *Solve* : $p=\tan\left(x-\frac{p}{1+p^2}\right)$.

On solving for x we get

$$x=\tan^{-1}p+\frac{p}{1+p^2} \qquad \text{......(1)}$$

Differentiating with respect to y,

$$\frac{2p\ dp}{(1+p^2)^2}=dy.$$

Integrating, $$c-\frac{1}{1+p^2}=y. \qquad \text{......(2)}$$

Equations (1) and (2) constitute the complete primitive.

Ex. 10. *Solve* : $ayp^2+(2x-b)\,p-y=0.$ (*Andhra* 1950)

Solve for x, $2x=\frac{y}{p}-ayp+b.$

Differentiating with respect to y

$$\frac{2}{p}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-ap-ay\frac{dp}{dy}$$

whence $$\left(p+y\frac{dp}{dy}\right)\left(\frac{1}{p^2}+a\right)$$

Taking first factor $p+y\frac{dp}{dy}=0$

Integrating $$py=c. \qquad \text{......(1)}$$

Eliminating p between (1) and the original equation we get

$$ac^2/y+(2x-b)\,c/y-y=0$$

i.e. $$ac^2+(2x-b)\ \ c-y^2=0.$$ This is the solution.

EXERCISE IV (C)

Solve :

1. $x=py-p^2$.
2. $(2x-b)\,p=y-ayp^2$.
3. $x=y+a\log p$.
4. $p^2y+2px=y$.
5. $x\,(1+p^2)=1$.
6. $x^2=a^2\,(1+p^2)$. (*Allahabad* 1959)
7. $y^2\log y=x\ yp+p^2$.

4·4. Clairaut's Equation. The differential equation of the form

$$y=px+f(p) \qquad \ldots\ldots(1)$$

is known as **Clairaut's equation.**

To solve it, differentiate it with respect to x. We get

$$p=p+\{x+f'(p)\}\frac{dp}{dx}$$

whence $\qquad \dfrac{dp}{dx}=0$ or $x+f'(p)=0$

From the first equation we get

$$p=c. \qquad \ldots\ldots(2)$$

Elimination of p from (1) and (2) gives

$$y=cx+f(c)$$

which is the required solution.

If we eliminate p between equation (1) and the equation.

$$x+f'(p)=0$$

we shall get an equation involving no constant and which is not a particular case of the solution $y=cx+f(c)$. Such a solution will be considered in the next chapter.

EXERCISE IV (D)

Solve :

1. $y=px+a/p$.   2. $y=x\,p+p-p^3$.

3. $y=x\dfrac{dy}{dx}+a\,\dfrac{dy}{dx}\left(1-\dfrac{dy}{dx}\right)$ (*Aligarh* 1934)

4. $y=xp+(1+p^2)^{1/2}$. (*Nagpur* 1936)

5. $y=xp+\sqrt{\{b^2-a^2p^2\}}$

6. $(y-px)(p-1)=p$. (*Nagpur* 1951, 1961)

7. $xp^2-yp+a=0$

8. $y=p\,(x-b)+a/p$.

9. $y=x\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3$. (*Andhra* 1938)

10. $4yp^2+2xp-y=0$ (*Lucknow* 1961)

4·51. Equations homogeneous in x and y. Such an equation can be put in the form

$$F\left(\frac{dy}{dx},\ \frac{y}{x}\right)=0$$

(*i*) *If it is solvable for* $\dfrac{dy}{dx}$ *i. e. for* p, it is the case already discussed.

(*ii*) *If it is solvable for* y/x, it can be put in the form

$$y/x=f(p) \text{ i.e. } y=x\,f(p).$$

Differentiating with respect to x, we get

$$p=f(p)+xf'(p)\frac{dp}{dx}$$

whence $$\frac{dx}{x}=\frac{f'(p)\,dp}{p-f(p)}$$

which can be integrated as the variables are separable.

4·52. Equation of the form $y=xf(p)+\phi(p)$.

Differentiating the given equation with respect to x, we have

$$p=f(p)+xf'(p)\frac{dp}{dx}+\phi'(p)\frac{dp}{dx},$$

or $$\frac{dx}{dp}+\frac{xf'(p)}{f'(p)-p}=\frac{\phi'(p)}{p-f(p)} \qquad \text{......(1)}$$

which is a linear equation for x regarded as a function of p. The elimination of p between (1) and the original equation will give the primitive.

Ex. 11. *Solve :* $y=(1+p)x+p^2$.

The equation is of the above form. Hence differentiating with respect to x, we get

$$p=1+p+x\frac{dp}{dx}+2p\frac{dp}{dx}$$

or $$\frac{dx}{dp}+x=-2p$$

which is linear and its solution is

$$x=2(1-p)+c\,e^{-p} \qquad \text{......(1)}$$

Substituting the value of x in the original equation

$$y=2-p^2+c\,e^{-p}(1+p) \qquad \text{......(2)}$$

(1) and (2) constitute the solution of the given equation

Ex 12. *Solve :* $e^{3x}(p-1)+p^3e^{2y}=0$.

(*Raj.* 1959, *Alld.* 1960, *Bombay* 1961)

Substituting $v=e^y$ and $u=e^x$ and therefore $p=\frac{dy}{dx}=\frac{u}{v}\frac{dv}{du}$, the equation is reduced to

$$v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^3$$

which is of Clairaut's form, hence its integral is

$$v=cu+c^3$$

i.e. $e^y=c\,e^x+c^3$.

Ex. 13. *Solve* : $y=2p\,x+y^2\,p^3$. (*Delhi* 1960, *Raj.* 1955, 1960)

Multiplying by y we get

$$y^2=2yx\,\frac{dy}{dx}+y^3\left(\frac{dy}{dx}\right)^3$$

Putting $y^2=Y$ and therefore $2y\,dy=dY$, we get

$$Y=x\,\frac{dY}{dx}+\frac{1}{8}\left(\frac{dY}{dx}\right)^3$$

$$\therefore\quad y^2=c\,x+\tfrac{1}{8}\,c^3.$$

Ex. 14 *Solve* : $y=-xp+x^4\,p^2$. (*Nagpur* 1958, *Raj.* 1956 *Cal.* 1954)

We have, $y=-x\,\dfrac{dy}{dx}+x^4\left(\dfrac{dy}{dx}\right)^2$,

Putting $-\dfrac{1}{x}=X$, therefore $\dfrac{dx}{x^2}=dX$, we get

$$y=\frac{X^2}{X}\,\frac{dy}{dX}+\left(\frac{dy}{dX}\right)^2=X\,\frac{dy}{dX}+\left(\frac{dy}{dX}\right)^2$$

$$\therefore\quad y=cX+c^2=-\frac{c}{x}+c^2.$$

Ex. 15. *Solve* : $y-2xp+ayp^2=0$.

Substituting $y^2=v$ and therefore $2yp=\dfrac{dv}{dx}$, we get

$$v=x\frac{dv}{dx}-\frac{a}{4}\left(\frac{dv}{dx}\right)^2$$

which is Clairaut's form. Hence the integral is

$$v=cx-\tfrac{1}{4}a\,c^2\quad i.e.\quad y^2=cx-\tfrac{1}{4}a\,c^2.$$

Ex. 16. *Solve* : $x^2\,(y-px)=yp^2$.
(*Alld.* 1948, *Delhi* 1947, 1950, *Cal.* 1954,
U. P. E S. 1951, *Nagpur* 1957)

Substituting $x^2=u$ and $y^2=v$ and therefore

$$\frac{dv}{du}=\frac{2y}{2x}\frac{dy}{dx}=\frac{y}{x}\,\frac{dy}{dx}\quad\text{or}\quad p=\frac{dv}{du}\sqrt{\frac{u}{v}}$$

the original equation becomes

$$v=u\frac{dv}{du}+\left(\frac{dv}{du}\right)^2$$

which is Clairaut's form Hence the integral is

$$cu+c^2\quad i.e.\quad y^2=cx^2+c^2.$$

Ex. 17. *Solve* : $xy\,(y-px)=x+py.$

Substituting $x^2=u$, $y^2=v$ and therefore

$$\frac{dy}{dx}=p=\frac{x}{y}\,\frac{dv}{du}=\frac{dv}{du}\sqrt{\frac{u}{v}}$$

the original equation becomes

$$\frac{dv}{v-1}=\frac{du}{1+u}$$

Integrating, $\quad \dfrac{v-1}{1+u}=c$ whence $y^2-1=c\,(x^2+1)$.

Ex. 18. *Solve* : $xy^2\,(p^2+2)=2py^3+x^3.$ (*Nagpur* 1958)

On factorization it becomes

$$(x-yp)\,(2y^2-x^2-xyp)=0$$

whence the first factor gives its primitive

$$x^2-y^2=c.$$

Substituting $y=vx$ and therefore $p=v+x\dfrac{dv}{dx}$, the second factor when equated to zero becomes

$$\frac{dx}{x}-\frac{v}{v^2-1}\,dv=0$$

Integrating, $\log x-\frac{1}{2}\log\,(v^2-1)=$constant *i.e.* $cx^2=(v^2-1)$

whence $\quad cx^4+x^2-y^2=0$, is the solution.

Hence the complete primitive is

$$(x^2-y^2+c)\,(x^2-y^2+cx^4)=0.$$

Ex. 19. *Solve* : $3p^2\,y^2-2xyp+4y^2-x^2=0.$

The equation can be written in the form

$$9p^2y^2-6xyp+12y^2-3x^2=0$$

i.e.

$$(x-3py)^2+12y^2-4x^2=0$$

$$(x-3py)^2-4\,(x^2-3y^2)=0$$

Substituting $x^2-3y^2=v^2$ and therefore

$$x-3y\,p=v\,\frac{dv}{dx}$$

the equation becomes

$$v^2\left(\frac{dv}{dx}\right)^2-4v^2=0 \text{ i.e. } \frac{dv}{dx}=2$$

whence $v=2x+c$ or $x^2-3y^2=4x^2+4xc+c^2$

Hence the primitive is

$$3x^2+3y^2+4cx+c^2=0.$$

Ex. 20. *Solve* : $\dfrac{(y-xp)}{\sqrt{(1+p^2)}}=f\,(x^2+y^2)^{\frac{1}{2}}$.

Substituting $x=r\cos\theta$. $y=r\sin\theta$, the given equation becomes

$$\frac{-r^2}{\sqrt{[\,r^2+(dr/d\theta)^2\,]}}=f\,(r)$$

whence
$$\frac{dr}{d\theta}=\frac{r\sqrt{[\,r^2-\{f\,(r)\}^2]}}{f\,(r)}$$

Therefore
$$\theta=\int\frac{f\,(r)\,dr}{r\sqrt{[\,r^2-\{f\,(r)\}^2]}}+c$$

where $\theta=\tan^{-1}y/x$ and $r=\sqrt{(x^2+y^2)}$.

Ex. 21. *Solve* : $y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=f\,\left(x+y\frac{dy}{dx}\right)$.

Differentiating with respect to x, the equation is reduced to

$$\left\{\frac{(dy/dx)}{\sqrt{\{1+(dy/dx)^2\}}}-f\,'\left(x+y\frac{dy}{dx}\right)\right\}\left\{1+\left(\frac{dy}{dx}\right)^2+y\,\frac{d^2y}{dx^2}\right\}=0$$

The second factor gives

$$1+\left(\frac{dy}{dx}\right)^2+y\,\frac{d^2y}{dx^2}=0$$

which on integration gives

$$x+y\frac{dy}{dx}=a,$$

or
$$(x-a)\,dx+y\,dy=0$$

Integrating again
$$(x-a)^2+y^2=b^2$$

which is the primitive. The first factor when equated to zero constitutes the differential equation of the singular solution which will be discussed in the next chapter.

Ex. 22. *Solve* : $(py+nx)^2=(y^2+nx^2)\,(1+p^2)$.

Substituting $y=vx$, and therefore

$$p=x\frac{dv}{dx}+v, \text{ we have}$$

$$x\frac{dv}{dx}=\pm\sqrt{\frac{n-1}{n}}\sqrt{(v^2+n)}$$

or $$\frac{dv}{\sqrt{(v^2+n)}}=\pm\sqrt{\left(\frac{n-1}{n}\right)}\cdot\frac{dx}{x}$$

On integration we get

$$\log[v+\sqrt{(v^2+n)}]=\pm\sqrt{\left(\frac{n-1}{n}\right)}\log x+\text{cons.}$$

therefore $$v+\sqrt{(v^2+n)}=c\,e^{\pm\sqrt{(n-1)/n}}$$

which on replacing v for y/x becomes

$$y+\sqrt{(y^2+nx^2)}=c\,x^{1\pm\sqrt{(n-1)/n}}$$

This is the complete primitive.

Ex. 23. *Solve* : $\left(1-y^2+\frac{y^4}{x^2}\right)p^2-2\frac{y}{x}p+\frac{y^2}{x^2}=0.$

(*Agra* 1962 ; *Jaipur* 1953, 1957 ; *Mysore* 1949)

We have $\left(p-\frac{y}{x}\right)^2=p^2y^2\left(1-\frac{y^2}{x^2}\right)^2$

Putting $y=vx$, $\therefore$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$, we get

$$x^2\left(\frac{dv}{dx}\right)^2=\left(\frac{dy}{dx}\right)^2v^2x^2(1-v^2)$$

or $$\frac{dv}{v\sqrt{1-v^2}}=\pm\,dy.$$

Putting $v=1/z$ on the L. H. S. we get on integration

$$\pm y+c=-\int\frac{dz}{\sqrt{z^2-1}}=-[\log z+\sqrt{z^2-1}\,]$$

$$=-\left[\log\frac{1}{v}+\sqrt{\frac{1}{v^2}-1}\right]$$

$$=\log\frac{y}{x+\sqrt{y^2-x^2}}.$$

Ex. 24. *Solve* :
$(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0.$

(*Rajasthan* 1954)

Dividing by $(1+p)^2$, the above equation becomes

$$(x^2+y^2)-2(x+y)\frac{(x+yp)}{1+p}+\left(\frac{x+yp}{1+p}\right)^2=0.$$

Now substitute $x^2+y^2=v$, $x+y=u$ and therefore

$$\frac{2(x+yp)}{1+p}=\frac{dv}{du}, \qquad \text{(on differentiation)}$$

(1) becomes $\qquad v=u\dfrac{dv}{du}-\dfrac{1}{4}\left(\dfrac{dv}{du}\right)^2$

which is Clairaut's form and hence

$$v=uc'-\tfrac{1}{4}c'^2$$

or $\qquad v+2uc+c^2=0, \qquad$ where $-\frac{1}{2}c'=c$

i.e. $\qquad x^2+y^2+2c(x+y)+c^2=0.$

EXERCISE IV (E)

Solve :

1. $yp^2+2xp-y=0$ (*Bombay* 1936)
2. $x+p/(1+p^2)^{1/2}=a.$ (*Nagpur* 1938)
3. $x^2p^2-2xyp+2y^2=x^2.$ (*Gorakhpur* 1959, *Sagar* 1950., *Cal. Hon's.* 1961)
4. $y=xp+x\sqrt{(1+p^2)}.$ (*Aligarh* 1937)
5. $x+py(2p^2+3)=0.$
6. $y=\dfrac{2ap^2}{(p^2+1)^2}.$
7. $(xp-y)^2=a(1+p^2)(x^2+y^2)^{3/2}.$
8. $4(xp^2+yp)=y^4.$
9. $2p^3-(2x+4\sin x-\cos x)p^2-(x\cos x-4x\sin x+\sin 2x)p+x\sin 2x=0.$
10. $(xp-y)^2=p^2-2\dfrac{y}{x}p+1.$ (*Lucknow* 1949)
11. $y-xp=x+yp.$
12. $a^2yp^2-4xp+y=0.$
13. $x^2(y-px)=yp^2.$
14. $\left(p^2-\dfrac{1}{a^2-x^2}\right)\left(p-\sqrt{\dfrac{y}{x}}\right)=0.$
15. $p^2(x^2-a^2)-2pxy+y^2+a^4=0.$ (*Patna* 1933)
16. $x+yp=ap^2.$
17. $xyp^2+p(3x^2-2y^2)-6xy=0.$ (*Madras* 1937)
18. $2y=xp+\dfrac{a}{p}.$

19. $y=ap+\sqrt{(1+p^2)}$
20. $(ap^2-b)\ xy+(bx^2-ay^2+c)\ p=0.$
21. $y=ap+bp^2.$
22. $p^3-(y+2x-e^{x-y})\ p^2+(2xy-2x\ e^{x-y}-y\ e^{x-y})\ p+2xy\ e^{x-y}=0.$
23. $(1+6y^2-3x^2y)\ p=3xy^2-x^2.$
24. $(1+x^2)\ p^2-2xyp+y^2=1.$
25. $(x^3y^3+x^2y^2+xy+1)\ y+(x^3y^3-x^2y^2-xy+1)\ xp=0.$ (*Allahabad* 1959)
26. $y-2px=f\ (x\ p^2).$
27. $x^2-\dfrac{xy}{p}=f\ (y^2-xyp).$
28. $\dfrac{2y}{x}-p=f\left(\dfrac{p}{x}-\dfrac{y}{x^2}\right).$
29. $\left(x\cos\dfrac{y}{x}+y\sin\dfrac{y}{x}\right)y=\left(y\sin\dfrac{y}{x}-x\cos\dfrac{y}{x}\right)xp.$
30. Use the transformation $u=x^2$ and $v=y^2$ to solve

$$(px-y)\ (py+x)=h^2p.$$

(*Allahabad* 1959, *Gorakhpur* 1959, *Sagar* 1959)

CHAPTER V

SINGULAR SOLUTIONS

5·1. Sometimes a solution of a differential equation can be found without involving any arbitrary constant and which is in general, not a particular case of the general solution. Such a solution we get when we solve the Clairaut's equation. We will discuss these solutions here.

Let $f(x, y, p)=0$ be the differential equation whose solution be $\phi(x, y, c)=0$ where c is an arbitrary constant. The c-discriminant is obtained by eliminating c between the equations

$$\phi(x, y, c)=0 \text{ and } \frac{\partial \phi}{\partial c}=0. \tag{1}$$

The p-discriminant is obtained by eliminating p between the equations

$$f(x, y, p)=0 \text{ and } \frac{\partial f}{\partial p}=0 \tag{2}$$

Evidently the c-discriminant is the locus for each point of which $\phi(x, y, c)=0$ has equal values of c and p-discriminant is the locus for each point of which $f(x, y, p)=0$ has equal values of p.

Now the *envelope* is the locus of points of intersection of the consecutive curves of the system obtained by giving different values to c in $\phi(x, y, c)=0$, and is obtained by eliminating c between the equations (1), and is therefore contained in the c-discriminant. Also the envelope is touched at any point on it by some curve of the system and therefore its x, y, p at each point are identical with x, y, p of some point on one of the curves of the system ; hence the equation to the envelope is also a solution of the differential equation $f(x, y, p)=0$.

Further at ultimate points of intersection of consecutive curves of ths system, p's of the intersecting curves are equal and locus of points when p's are equal will include the envelope, that is, the equation of the envelope is also contained in the p-disc. Thus both the c-disc. and the p-disc. contain the equation to the envelope. This is called the *singular solution* It evidently satisfies the differential equation but is not contained in the general solution and is not obtained by giving particular values to the constant in the general solution.

The p–discriminant gives equal values of p but these values may belong to two curves of the system that are not consecutive, that is, these curves have different c's. Locus of such points is called the *tac-locus*. For example, consider a family of circles, all of equal radii, whose centres lie on a straight line.

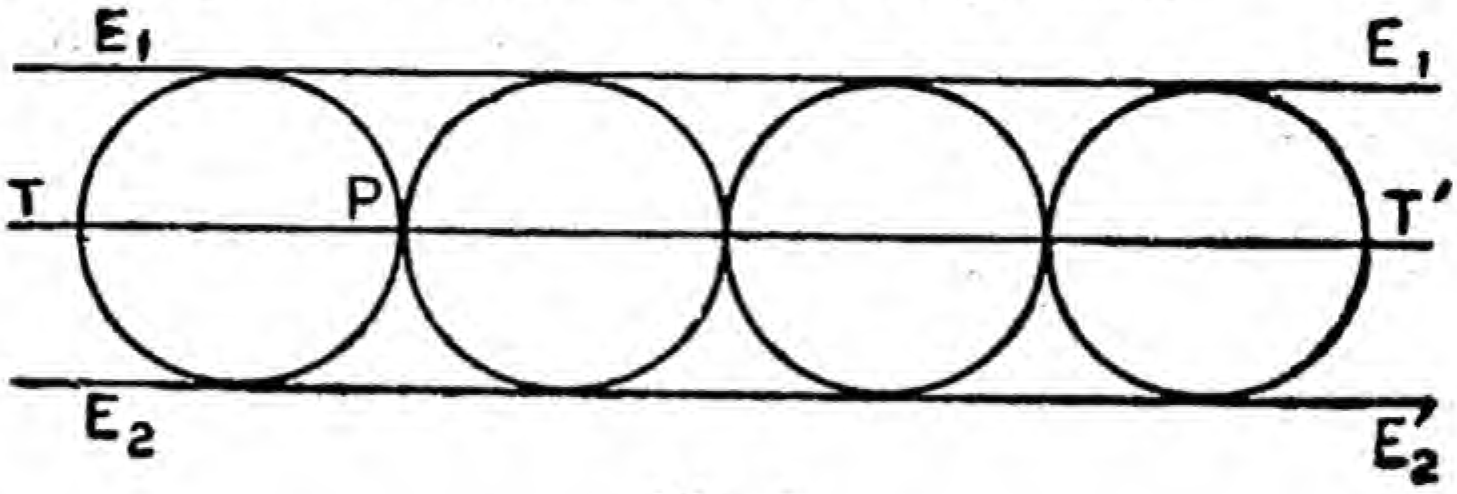

Fig. 1

At a point such as P, the two circles, which are not consecutive, touch, that is, have same p but the direction of the tangent at P is not the direction of the line of centres which is the locus of the points of contact of the circles. The line of centres is the tac-locus.

The c-disc. gives equal values of c but these values may belong to the nodes which are also ultimate points of intersection of the consecutive curves. Locus of such points is called the *nodal-locus*. The p for the nodal locus may be different from p of the curve passing through the node hence the x, y, p belonging to the nodal locus will not satisfy the differential equation as in Fig 2. In some cases however they may satisfy the differential equation, in which cases the nodal locus would also be an envelope as in Fig. 3.

Fig. 2

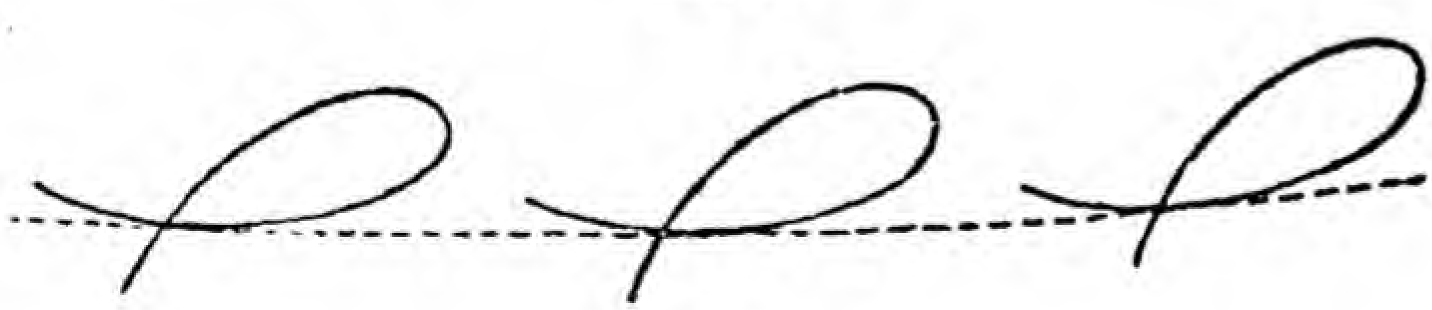

Fig. 3

As in the case of nodes, the cusps will also appear in the c-disc., as they are also the ultimate points of intersection of the consecutive curves. Their locus is called the *cuspidal-locus*. But for a cusp, p's are equal, that is, the cuspidal locus will also be contained in the p-disc. The cuspidal locus is not in general a solution of the differential equation as in Fig. 4 but in some exceptional case it may, as in Fig. 5.

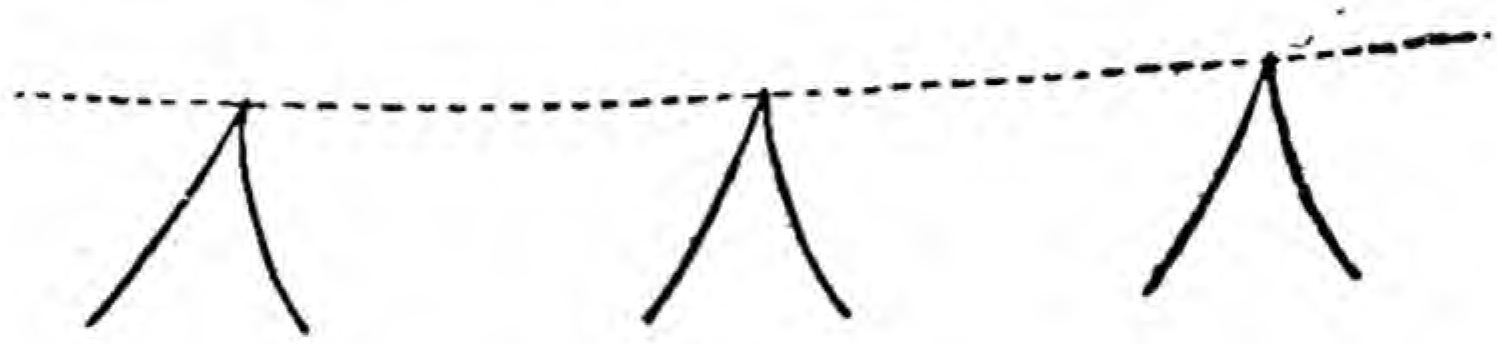

Fig. 4

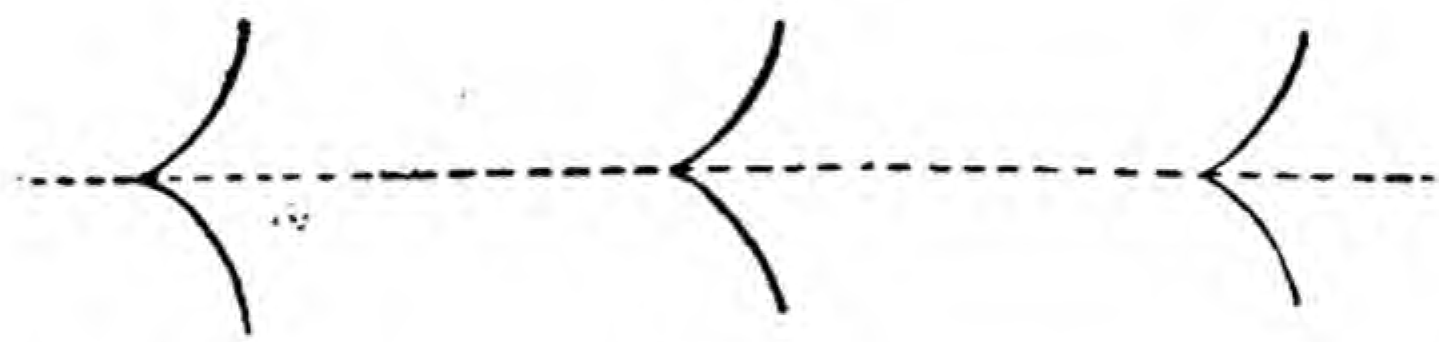

Fig. 5

It may be added that the locus, obtained from c-disc. contains the envelope as a factor once, the nodal-locus twice and the cuspidal-locus thrice and that the locus, obtained from p-disc. contains the envelope as a factor once, the cuspidal-locus once, and the tac-locus twice.

Thus

$$c\text{-discriminant} \sim EN^2\ C^3 = 0$$
$$p\text{-discriminant} \sim ET^2\ C = 0.$$

We have seen that the singular solution, that is, the envelope is obtained as the common factor from the c-and p-discriminants and that it must satisfy the differential equation. We now find the analytical condition for this: Suppose by solving the second equation in (2) for p and substituting this value of p in the first equation we get $$f' = 0 \qquad \ldots\ldots(3)$$

If this equation is a solution of the differential equation then the p obtained from differentiating this equation will be the same p as in f. Hence from (3) we have

$$\frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial y}\, p = 0.$$

But $f'(x, y)=f(x, y, p)$ so that

$$\frac{\partial f'}{\partial x}+\frac{\partial f'}{\partial y}p=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p+\frac{\partial f}{\partial p}\frac{dp}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p$$

because $\frac{\partial f}{\partial p}=0$, hence p is given by

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0.$$

Thus we conclude that if a singular solution of the differential equation $f(x, y, p)=0$ exists, it must simultaneously satisfy the equations

$$f=0,\ \frac{\partial f}{\partial p}=0,\ \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}p=0. \qquad \ldots\ldots(4)$$

If however, the value of p is infinite then the last equation gives $\frac{\partial f}{\partial y}=0$. In that case it is better to verify if the solution satisfies the differential equation.

5·2. Solved Examples.

Ex. 1. *Obtain the primitive and the singular solution of the equation*

$$p^2(1-x^2)=1-y^2.$$

Specify the nature of the geometrical loci which are not singular solutions, but which may be obtained with the singular solutions.

(*Agra* 1961)

The given equation is

$$p^2=\frac{1-y^2}{1-x^2} \text{ or } \left(\frac{dy}{dx}\right)^2=\frac{1-y^2}{1-x^2}$$

or $$\frac{dy}{\sqrt{1-y^2}}-\frac{dx}{\sqrt{1-x^2}}=0,$$

or $$\sin^{-1}y-\sin^{-1}x=\text{constant}$$

or $$\cos(\sin^{-1}y-\sin^{-1}x)=\text{constant}.$$

or $$\sqrt{1-y^2}\sqrt{1-x^2}+yx=A \text{ (say)}$$

or $$(1-y^2)(1-x^2)=(A-yx)^2$$

or $$1-y^2-x^2+y^2x^2=A^2-2A\,yx+y^2x^2$$

Thus the primitive is the family of conics

$$x^2+y^2-2A\,xy=1-A^2.$$

The c–discriminant is given by

$$A^2-2A\,xy+x^2+y^2-1=0$$

which is
$$4x^2y^2-4x^2-4y^2+4=0$$

i.e.
$$(x^2-1)\,(y^2-1)=0$$

The p–discriminant is given by

$$p^2\,(1-x^2)-(1-y^2)=0,$$

i.e.
$$4\,(1-y^2)\,(1-x^2)=0.$$

The lines $y=\pm 1$ and $x=\pm 1$ are all touched by the family of conics. These lines constitute the singular solution.

Ex. 2. *Examine the equation $y^2\,(1+p^2)=r^2$ for singular solution.*

Solving for p, $p=\dfrac{\sqrt{r^2-y^2}}{y}=\dfrac{dy}{dx}$

or $\quad dx=\dfrac{y}{\sqrt{r^2-y^2}}\,dy$

Integrating, $\quad x+c=-\sqrt{r^2-y^2}\quad$ or $\quad y^2+(x+c)^2=r^2,$

c–discriminant is $y^2-r^2=0$

p–discriminant is $y^2\,(y^2-r^2)=0.$

The equation $y^2-r^2=0$ occurs in both c–and p–discriminants and also satisfies the differential equation, it is therefore the singular solution.

The equation $y=0$ does not satisfy the differential equation but appears twice as a factor in p–disc. and is therefore the tac-locus. In Fig. 1, $E_1\,E_1'$ and $E_2\,E_2'$ are the envelopes and TT' is the tac-locus.

Ex. 3. *Find the complete primitive and the singular solution of the differential equation*

$$\sin\left(x\frac{dy}{dx}\right)\cos y=\cos\left(x\frac{dy}{dx}\right)\sin y+\frac{dy}{dx} \qquad \textit{(Jaipur 1953)}$$

The given equation can be written in the form.

$$\sin\left(x\,\frac{dy}{dx}-y\right)=\frac{dy}{dx}$$

or
$$xp-y=\sin^{-1} p$$

or
$$y=px-\sin^{-1} p \qquad \ldots\ldots(1)$$

which is Clairaut's form and its solution is

$$y=cx-\sin^{-1} c \qquad \ldots\ldots(2)$$

Differentiating (1) with respect to p

$$0=x-\frac{1}{\sqrt{(1-p^2)}} \text{ giving } p=\frac{\sqrt{(x^2-1)}}{x} \qquad \ldots\ldots(3)$$

Eliminating p between (1) and (3), the p-discriminant is

$$y=\sqrt{(x^2-1)}-\sin^{-1}\frac{\sqrt{(x^2-1)}}{x} \qquad \ldots\ldots(4)$$

Similarly c-discrimnant is also the same.

Hence equation (4) is the singular solution.

Ex. 4. *Examine the equation* $4xp^2=(3x-a)^2$ *for singular solution.* (*Agra* 1952)

$$\frac{dy}{dx}=\frac{3x-a}{2\sqrt{x}} \text{ or } dy=\left(\frac{3}{2}\sqrt{x}-\frac{a}{2\sqrt{x}}\right)dx$$

Solution is $y+c=x^{3/2}-a\,x^{1/2}$ or $(y+c)^2=x\,(x-a)^2$.

$\therefore$ c-discriminant is $x\,(x-a)^2=0$,

p-discriminant is $x\,(3x-a)^2=0$.

$\therefore$ $x=0$, which is a solution of the differential equation, is the singular solution ; the equation $x=a$, which occurs twice in c-discriminant is the nodal locus and $x=\frac{1}{3}a$ which occurs twice in p-discriminant is the tac-locus.

Ex. 5. *Obtain the primitive and singular solutions of the following equation :*

$$4p^2\,x(x-a)(x-b)=\{3x^2-2x\,(a+b)+ab\}^2,$$

Specify the nature of the loci which are not solutions but which are obtained with the singular solution. (*Agra* 1651, 1960)

Applying the condition that p should have equal roots

$$16\,x\,(x-a)\,(x-b)\,\{3x^2-2\,(a+b)\,x+ab\}^2=0 \qquad \ldots\ldots(1)$$

Also $$4p^2=\frac{\{3x^2-2\,(a+b)\,x+ab\}^2}{x\,(x-a)\,(x-b)}$$

or $$2p=\frac{3x^2-2\,(a+b)\,x+ab}{\sqrt{x\,(x-a)\,(x-b)}}=\frac{3x^2-2\,(a+b)\,x+ab}{\sqrt{x^3-(a+b)x^2+abx}}$$

or $$\frac{dy}{dx}=\frac{1}{2}\,\frac{3x^2-2\,(a+b)\,x+ab}{\sqrt{x^3-(a+b)x^2+ab\,x}}$$

or $$y+c=\sqrt{x^3-(a+b)x^2+ab\,x}=\sqrt{x\,(x-a)\,(x-b)}.$$

Hence the primitive is $(y+c)^2=x\,(x-a)\,(x-b)$,

or $$c^2+2cy+y^2-x\,(x-a)\,(x-b)=0$$

The condition that c will have equal values is

$$4y^2-4\,\{\,y^2-x\,(x-a)\,(x-b)\,\}=0$$

or $$x\,(x-a)\,(x-b)=0. \qquad \ldots\ldots(2)$$

In (1) and (2) $x(x-a)(x-b)$ are the common factors and the differential equation is satisfied by $x=0, x=a, x=b$. Hence these are the singular solutions. The remaining factor in the p-discriminant is $3x^2-2x(a+b)+ab=0$ which gives

$$x=\frac{2(a+b)\pm\sqrt{4(a+b)^2-12ab}}{6}$$

or $$3x=(a+b)\pm(a^2-ab+b^2)^{1/2}$$

Since this is a square factor, hence these lines are tac-loci.

The curve $y^2=x(x-a)(x-b)$, $(0<a<b)$ consists of an oval cutting the axis of x at the origin, and at distance a, and of a curve like a parabola cutting the axis of x at a distance b. The tangents at all these points are parallel to the axis of y which are

$$x=0,\ x=a,\ x=b.$$

The system of curves is obtained by moving this curve parallel to the axis of y. The straight lines $x=0$, $x=a$, $x=b$ are envelopes

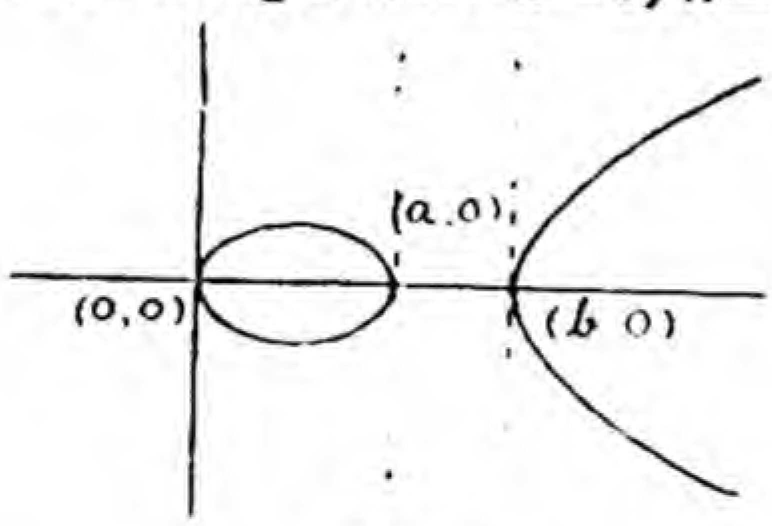

Fig. 6

of the system. The line $3x=a+b-\sqrt{(a^2-ab+b^2)}$ is a tac-locus of real points of contact; the line

$$3x=a+b+\sqrt{(a^2-ab+b^2)}$$

is a tac-locus of imaginary points of contact.

Ex. 6. *Solve and test for singular solutions*

$$p^3-4xy\,p+8y^2=0$$

(*I.A.S.* 1951, *Agra* 1955, *Jaipur* 1957)

Substituting $y=z^2$ i.e. $p=\frac{dy}{dx}=2z\frac{dz}{dx}$, we get

$$8z^3\left(\frac{dz}{dx}\right)^3-4x.\,z^2.\,2z\frac{dz}{dx}+8z^4=0$$

or $$z=x\frac{dz}{dx}-\left(\frac{dz}{dx}\right)^3$$

$\therefore$ The solution is $z=xc'-c'^3$

or $\sqrt{y}=xc'-c'^3$

or $y=c'^2(x-c'^2)^2=c(x-c)^2.$

Now differentiating the given equation

$$p^3-4xy\,p+8y^2=0 \qquad \ldots\ldots(1)$$

partially with respect to p, we get

$$3p^2-4xy=0 \qquad \ldots\ldots(2)$$

Eliminating p between (1) and (2) we get $y(27y-4x^3)=0$ which is the p-discriminant.

Also differentiating $y=c(x-c)^2 \qquad \ldots\ldots(3)$

partially with respect to c we get, $-2c(x-c)+(x-c)^2=0$

or $(x-c)(x-3c)=0 \qquad \ldots\ldots(4)$

Eliminating c between (3) and (4) we get

if $c=x$, $y=0$ and if $c=\frac{1}{3}x$, $27y-4x^3=0$,

$\therefore$ c-discriminant is $y(27y-4x^3)=0$:

Hence $y=0$ and $27y-4x^3=0$ are the singular solutions.

Also clearly $y=0$ is a particular integral of the equation.

Ex. 7. *Examine the equation $p^2+2xp-y=0$ for singular solution.*

Here $(p+x)^2=y+x^2$ or $p=\sqrt{y+x^2}-x$

Put $y=x^2v$ $\therefore$ $x^2\dfrac{dv}{dx}+2xv=x(\sqrt{1+v}-1)$

or $$x\frac{dv}{dx}=\sqrt{1+v}-1-2v$$

or $$\frac{dx}{x}=\frac{dv}{\sqrt{1+v}-1-2v}, \qquad \text{put } 1+v=u^2$$

$$=\frac{2u\,du}{1+u-2u^2}=-\frac{1}{2}\,\frac{(1-4u)\,du-du}{1+u-2u^2}$$

Integrating we get

$$\log x=-\tfrac{1}{2}\log(1+u-2u^2)+\tfrac{1}{2}\int\frac{du}{1+u-2u^2}+\text{constant}$$

$$=-\tfrac{1}{2}\log(1+u-2u^2)+\tfrac{1}{6}\log\frac{2(1-u)}{1+2u}+\text{constant}$$

Hence $x^6(1+u-2u^2)^3\dfrac{(1-u)}{1+2u}=\text{constant}$

or $$x^6 (1-u)^4 (1+2u)^2 = \text{constant}$$
or $$x^3 (1-u)^2 (1+2u) = c.$$

or $$1-3u^2+2u^3 = \frac{c}{x^3}$$

or $$1-3\left(1+\frac{y}{x^2}\right)+2u\left(1+\frac{y}{x^2}\right) = \frac{c}{x^3}$$

giving finally $(2x^3+3xy+c)^2 = 4(x^2+y)^3$

p-discriminant is $x^2+y=0$,
c-discriminant is $(x^2+y)^3=0$.

Hence $x^2+y=0$ is the cuspidal locus.

Ex. 8. *Examine the equation $x^3p^2+x^2yp+a^3=0$ for singular solution. Find also the tac-locus.*

Here $y=-xp-\dfrac{a^3}{x^2 p}$.

Differentiating we get

$$p=-p-x\frac{dp}{dx}+\frac{a^3\left(2xp+x^2\frac{dp}{dx}\right)}{x^4 p^2}$$

or $$2p+x\frac{dp}{dx}=\frac{a^3 x\left(2p+x\frac{dp}{dx}\right)}{x^4 p^2}$$

Hence $2p+x\dfrac{dp}{dx}=0$ giving $p=\dfrac{c}{x^2}$.

$\therefore$ Solution is $a^3x+cxy+c^2=0$

p-discriminant is $x^3(xy^2-4a^3)=0$.
c-discriminant is $x(xy^2-4a^3)=0$.

$\therefore$ Singular solution is $x(xy^2-4a^3)=0$,
and tac-locus is $x=0$.

Ex. 9. *Reduce the differential equation*

$$x^2p^2+yp(2x+y)+y^2=0, \text{ where } p=\frac{dy}{dx}$$

to Clairaut's form by the substitution $Y=y$, $X=xy$. Hence, or otherwise, solve the equation. Prove that $y+4x=0$ is a singular solution : and that $y=0$ is both part of the envelope and part of an ordinary solution.

(*Bombay* 1961, *Jaipur* 1956, 1959, *Agra* 1957, 1962)

Put $y=Y,\ xy=X$ so that $y=Y,\ x=\dfrac{X}{Y}$

hence $dy=dY,\ dx=\dfrac{1}{Y}\ dX-\dfrac{X}{Y^2}\,dY$

hence $p=\dfrac{dY}{\dfrac{1}{Y}\ dX-\dfrac{X}{Y^2}\,dX}=\dfrac{Y^2\,P}{Y-XP}$ where $P=\dfrac{dY}{dX}$

The differential equation is $(xp+y)^2+y^3\,p=0.$

$$\therefore\ \left(\frac{X}{Y}\cdot\frac{X^2\,P}{Y-XP}+Y\right)^2+Y^2\,\frac{Y^2\,P}{Y-XP}=0$$

which simplifies into $Y=XP-\dfrac{1}{P}.$

$\therefore$ Solution is $Y=cX-\dfrac{1}{c}$, i.e. $y=cxy-\dfrac{1}{c}$

or $c^2xy-cy-1=0$

c-disc. is $y^2+4xy=0$ *i.e.* $y(y+4x)=0$

p-disc. is $y^3(y+4x)=0.$

$\therefore$ Singular solution is $y\ (y+4x)=0.$

Hence the result.

Ex. 10. *Examine the equation $xp^2-2yp+x+2y=0$ for singular solution. Hence interpret the primitive and singular solution.* (*Agra* 1959)

Put $y-x=Y,\ x^2=X$ so that $x=\sqrt{X},\ y=Y+\sqrt{X},$

Hence $dy=dY+\dfrac{1}{2\sqrt{X}}\,dX,\ dx=\dfrac{1}{2\sqrt{X}}\,dX$

$\therefore\ p=2\sqrt{X}\,P+1$ where $P=\dfrac{dY}{dX}$

The equation is $x(1+p^2)=2y\ (p-1)$

which reduces to $Y=XP+\dfrac{1}{2P}$

$\therefore$ Solution is $Y=cX+\dfrac{1}{2c}$ or $y-x=cx^2+\dfrac{1}{2c}$

or $2c^2\,x^2-2c\ (y-x)+1=0$ ......(1)

c-disc. is $(y-x)^2=2x^2.$ ......(2)

p-disc. is also the same hence it is the signular solution.

Clearly (1) represents a family of parabolas and (2) represents a pair of straight lines $y-x=\pm\sqrt{2}x$. Hence eq^n represents a parabola touching a pair of straight lines.

Ex. 11. *Obtain the singular solution of the equation*

$$p^2 y^2 \cos^2 \alpha-2pxy \sin^2 \alpha+y^2-x^2 \sin^2\alpha=0$$

directly from the equation and also from its complete primitive, explaining the geometrical significance of the irrelevant factors that present themselves. (*Nagpnr* 1961, *Jaipur* 1950, *Agra* 1953)

The condition that the equation

$$p^2 y^2 \cos^2 \alpha-2pxy \sin^2 \alpha+y^2-x^2 \sin^2 \alpha=0$$

may have equal roots in p is

$$x^2 y^2 \sin^4 \alpha=y^2 \cos^2 \alpha\ (y^2-x^2 \sin^2 \alpha),$$

i.e. $\quad (x^2 \sin^2 \alpha-y^2 \cos^2 \alpha)\ y^2=0$

so that $y=0$ and $y=\pm x \tan \alpha$.

Now the given equation is

$$p^2 y^2-2pxy \tan^2 \alpha+y^2 \sec^2 \alpha-x^2 \tan^2 \alpha=0.$$

Solving $\quad py=x \tan^2 \alpha\pm\sec \alpha\sqrt{x^2 \tan^2 \alpha-y^2}$

or $\quad \pm\dfrac{y\,dy-x \tan^2 \alpha\, dx}{\sqrt{x^2 \tan^2 \alpha-y^2}}=\sec \alpha\, dx$

or $\quad \therefore\ \pm\sqrt{x^2 \tan^2 \alpha-y^2}=c-x \sec \alpha$

or $\quad x^2+y^2-2cx \sec \alpha+c^2=0$

Hence c–discriminant is given by

$$x^2 \tan^2 \alpha-y^2=0 \quad \text{or} \quad y=\pm x \tan \alpha.$$

Also, the complete primitive can be written as

$$c^2-2cx \sec \alpha+x^2 \sec^2 \alpha=x^2 \tan^2 \alpha-y^2$$

or $\quad x^2+y^2-2cx \sec \alpha+c^2=0$

or $\quad (x-c \sec \alpha)^2+y^2=c^2 \tan^2 \alpha$

Thus the curve represents a series of circles, their envelope is given by the two straight lines $y=\pm x \tan \alpha$.

Since they are common both in p-discriminant and also in c-discriminant, hence they constitute the singular solution.

The line $y=0$ is a tac-locus.

Ex. 12. *Transform the equation*

$$(2x^2+1)\left(\frac{dy}{dx}\right)^2+(x^2+2xy+y^2+2)\frac{dy}{dx}+2y^2+1=0$$

to Clairaut's form by putting $x+y=u$, $xy-1=v$ and integrate it.

Here $\quad x+y=u$, therefore $dx+dy=du$

$\quad xy-1=v$, therefore $xdy+ydx=dv$.

$$\therefore \quad P=\frac{dv}{du}=\frac{x\,dy+y\,dx}{dy+dx}=\frac{xp+y}{1+p} \text{ where } p=\frac{dy}{dx}.$$

Solving it $p=(y-P)/(P-x)$.

Substituting this value in the given equation

$$(2x^2+1)\left[\frac{y-P}{P-x}\right]^2+(x^2+2xy+y^2+2)\,\frac{y-P}{P-x}+2y^2+1=0$$

$$(2x^2+1)\,(y-P)^2+(x^2+2xy+y^2+2)\,(y-P)\,(P-x)$$
$$+(2y^2+1)\,(P-x)^2=0$$

$$P^2\,(x^2+y^2-2xy)+P\,\{\,(x+y)\,(x^2+y^2-2xy)\,\}$$
$$-(xy-1)\,(x^2+y^2-2xy)=0$$

or $\qquad P^2+P\,(x+y)-(xy-1)=0$

or $\qquad v=u\,P+P^2$, which is the Clairaut's form.

$\therefore$ The solution is

$$v=u\,c+c^2$$

or $\qquad xy-1=(x+y)\,c+c^2.$

EXERCISE V

Examine for the singular solution :

1. $x^2\,y^2-3xyp+2y^2+x^3=0.$ (Put $x=X$, $y/x=X$)
2. $xp^2-2yp+ax=0.$
3. $y^2-2pxy+p^2\,(x^2-1)=m^2.$ (*Punjab* 1956)
4. $y=xp-p^2.$
5. $4p^3=9x.$
6. $4x\,(x-1)\,(x-2)\,p^2-(3x^2-6x+2)^2=0.$ (*Agra* 1956)
7. $(8p^3-27)x=12\,p^2y.$ (*L.I.C.* 1960)
8. $3y=2px-2\,p^2/x.$
9. $y^2+p^2=1.$
10. $p^2\,(2-3y)^2=4\,(1-y).$
11. $4xp^2=(3x-1)^2.$
12. $xp^2-(x-a)^2=0$ (*I A.S.* 1959)
13. $yp^2-2xp+y=0.$
14. $3xp^2-6yp+x+2y=0.$
15. $p^2+2px^3-4x^2y=0.$
16. $y^2\,(y-xp)=x^4\,p^2.$ (*Agra* 1934)
17. $(x^2-a^2)\,p^2-2xyp-x^2=0.$
18. $p^4=4y\,(xp-2y)^2.$
19. $(1-y^2)\,p^2=1.$
20. $x^2+y=p^2.$
21. $p^3=y^4\,(y+xp).$
22. $(1-p)^2-e^{-2y}=p^2\,e^{-2x}.$
23. $axyp^2+(x^2-ay^2-b)\,p-xy.$ (*I.A.S.* 1935 ; *Alld.*, 1960 ; *Agra* 1944)
24. $p^2=(4y+1)\,(p-y).$
25. $(a^2-x^2)\,p^2+2xyp+b^2-y^2=0.$ (*I. A. S.* 1957)
26. Reduce $xyp^2-(x^2+y^2-1)\,p+xy=0$ to Clairaut's form by the substitution $X=x^2$; $X=y^2$. Hence show that the equation represents a family of conics touching the four sides of a square. (*Raj.*, 1948, *Agra* 1958)
27. Show that $xyp^2+(x^2-y^2-h^2)\,p-xy=0$ represents a family of confocal conics, with the foci at $(\pm h, 0)$ touching the four imaginary lines joining the foci to the circular points at infinity. (*Agra* 1937)

28. Show that the complete primitive of $8p^3x=y(12p^2-9)$ is $(x+c)^3=3y^2c$, the p-discriminant is $y^2(9x^2-4y^2)=0$, and c-discriminant is $y^4(9x^2-4y^2)=0$. Interpret these discriminants. (*Punjab* 1956)

29. Show that the curves of the family $y^2-2cx^2y+c^2(x^4-x^3)=0$ all have a cusp at the origin, touching the axis of x.

By eliminating c obtain the differential equation of the family in the form $4p^2x^2(x-1)-4pxy(4x-3)+(16x-9)y^2=0$.

Show that both discriminants take the form $x^3y^2=0$, but that $x=0$ is not a solution, while $y=0$ is a particular integral.

30. Reduce to Clairaut's form the equation

$$(px^2+y^2)(px+y)=(p+1)^2$$

by substituting $u=xy$ and $v=x+y$.

Show also that $x^2y^2+4(x+y)=0$ is the envelope locus and $x-y=0$ is the tac-locus of the system given by its general solution. (*Rajasthan* 1955, 1959)

31. Show that the equations

$$y-xp=a(y^2+p) \text{ and } y-xp=b(1+x^2p)$$

are derivable from a common primitive. (*Lucknow* 1953)

32. Reduce the differential equation $(px-y)(x-py)=2p$ to Clairaut's form by the substitution $x^2=u$ and $y^2=v$ and find its complete primitive and its singular solution, if any. (*Raj* 1949, *I. A. S.* 1951, *Agra* 1954)

[**Hint.** Making the substitution $x^2=u$, $y^2=v$

$\therefore$ $2x\,dx=du$, $2y\,dy=dv$ we get

$$u\frac{dv}{du}-v=\frac{2dv/du}{1-dv/du}$$

This is Clairaut's form. Hence solution is

$$v=uc+\frac{2c}{c-1} \text{ or } c^2x^2+c(2-y^2-x^2)+y^2=0$$

The c-disc. is given by $(2-y^2-x^2)^2=4x^2y^2$

i.e. $(x+y+\sqrt{2})(x+y-\sqrt{2})(x-y+\sqrt{2})(x-y-\sqrt{2})=0$

The p discriminant is also the same].

CHAPTER VI

HOMOGENEOUS LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

6·1. The Homogeneous Linear Equation. *A linear differential equation of the form*

$$x^n \frac{d^n y}{dx^n}+A_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}}+\dots\dots+A_n y=X, \quad \dots(1)$$

where A_1, A_2,........., A_n are constants, and X is either a constant or a function of x is called a **homogeneous linear differential equation.**

To solve this type of equation we generally change the independent variable x to z by substituting $x=e^z$.

We have $\dfrac{dx}{dz}=e^z=x,$

$$\therefore \quad \frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=x\,\frac{dy}{dx}.$$

$$\therefore \quad x\,\frac{d}{dx}\equiv\frac{d}{dz}.$$

Let D stand for $\dfrac{d}{dz}$, then

$$x\,\frac{d}{dx}\left(x^{n-1}\frac{d^{n-1} y}{dx^{n-1}}\right)=x^n\frac{d^n y}{dx^n}+(n-1)\,x^{n-1}\frac{d^{n-1} y}{dx^{n-1}}$$

or

$$x^n\,\frac{d^n y}{dx^n}=\left(x\,\frac{d}{dx}-n+1\right)x^{n-1}\,\frac{d^{n-1} x}{dx^{n-1}}.$$

$$\therefore \quad \boxed{x^n\frac{d^n y}{dx^n}=(D-n+1)\,x^{n-1}\,\frac{d^{n-1} y}{dx^{n-1}}.}$$

Now putting $n=2, 3, 4$.........in this, we have

$$x^2\frac{d^2 y}{dx^2}=(D-1)\,x\frac{dy}{dx}=D\,(D-1)\,y,$$

$$x^3\frac{d^3 y}{dx^3}=(D-2)\;x^2\;\frac{d^2 y}{dx^2}=(D-2)\;(D-1)\;Dy, \text{ etc.}$$

$$\therefore \quad x^n \frac{d^n y}{dx^n} = (D-n+1)(D-n+2)\ldots\ldots(D-1)\, Dy$$

$$= D\,(D-1)\,(D-2)\ldots\ldots(D-n+1)\, y.$$

Thus the transformed equation is

$$[\, D\,(D-1)\ldots(D-n+1) + A_1\, D\,(D-1)\ldots(D-n+2)$$
$$+\ldots\ldots\ldots + A_n \,]\, y = Z \quad \ldots(2)$$

where Z is the function of z into which X is changed.

This is an equation with constant coefficients and therefore can be integrated by the methods previously discussed. If the transformed equation is

$$f(D)\; y = Z,$$

then $C.\ F.$ is given by the roots of $f(D)=0$.

If m_1, m_2 are the roots of $f(D)=0$, then

$$C.\ F. = c_1\, e^{m_1 z} + C_2\, e^{m_2 z} = c_1\; x^{m_1} + c_2\, x^{m_2}$$

If $m_1 = m_2$, the $C.\ F. = (c_1 + c_2\, z)\, e^{m_1 z}$

$$= (c_1 + c_2 \log x)\; x^{m_1}.$$

In case there are r roots each equal to m, the corresponding $C.\ F.$ is

$$x^m\, [\, c_1 + c_2 \log x + \ldots\ldots + c_r\, (\log x)^{r-1}\,]$$

If the roots are imaginary say $\alpha \pm i\beta$, then

$$C.\ F. = e^{\alpha z}\; (c_1 \cos \beta z + c_2 \sin \beta z)$$

$$= x^{\alpha}\; [\, c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\,]$$

Similarly the Particular Integral is given by

$$\frac{1}{f(D)}\; .\, Z.$$

Ex. 1. *Solve* :

$$x^4 \frac{d^3 y}{dx^3} + 2x^3\; \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1.$$

(*Bombay* 1961, *Agra* 1955, *Mysore* 1948)

Dividing both sides of the equation by x

$$x^3\; \frac{d^3 y}{dx^3} + 2x^2\; \frac{d^2 y}{dx^2} - x\; \frac{dy}{dx} + y = \frac{1}{x}\, .$$

On changing the independent variable by putting $x = e^z$, this equation is reduced to

$$[\, D\,(D-1)\,(D-2) + 2D\,(D-1) - D + 1\,]\; y = e^{-z},$$

or

$$(D-1)^2\,(D+1)\; y = e^{-z}.$$

The $C.F.=(c_1+c_2 z)\ e^{z}+c_3\ e^{-z}$
$=(c_1+c_2 \log x)\ x+c_3\ x^{-1}$

$$P.I.=\frac{1}{(D-1)^2\ (D+1)}\ .\ e^{-z}=\frac{1}{D+1}\cdot\frac{1}{(D-1)^2}\ e^{-z}$$

$$=\frac{1}{D+1}\ \frac{e^{-z}}{4}=\frac{e^{-z}}{4}\ \frac{1}{D}\ .\ 1=\frac{e^{-z}\ .\ z}{4}=\frac{x^{-1}\log x}{4}$$

Hence the complete integral is

$$y=(c_1+c_2 \log x)\ x+c_3\ x^{-1}+\tfrac{1}{4}\ x^{-1}\log x.$$

Ex. 2. *Solve* : $x^2\dfrac{d^2y}{dx^2}-2y=x^2+\dfrac{1}{x}$. (*Jaipur* 1951)

On changing the independent variable by putting $x=e^z$, the equation is reduced to $D\ (D-1)\ y-2y=e^{2z}+e^{-z}$

i.e. $(D^2-D-2)\ y=e^{2z}+e^{-z}$

The auxiliary equation is : $m^2-m-2=0$ giving $m=2$ or -1

$\therefore$ $C.\ F.$ is $c_1\ e^{2z}+c_2\ e^{-z}=c_1\ x^2+c_2\ x^{-1}$

$$P.\ I.=\frac{1}{D^2-D-2}\ (e^{2z}+e^{-z})$$

$$=\frac{1}{(D-2)(D+1)}\ e^{2z}+\frac{1}{(D-2)(D+1)}\ e^{-z}$$

$$=\frac{1}{3(D-2)}\ e^{2z}-\frac{1}{3\ (D+1)}\ e^{-z}$$

$$=\frac{1}{3}\ e^{2z}\ .\ \frac{1}{D}\ .\ 1-\frac{1}{3}\ e^{-z}\ .\ \frac{1}{D}\ .\ 1$$

$$=\tfrac{1}{3}\ z\ (e^{2z}-e^{-z})=\tfrac{1}{3}\log x\ (x^2-1/x)$$

Hence the complete integral is

$$y=c_1\ x^2+c_2\ x^{-1}+\tfrac{1}{3}\log x.(x^2-1/x)$$

Ex. 3. *Solve* : $x^3\dfrac{d^3y}{dx^3}-x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=x^2+3x$.

(*Raj.* 1955)

On changing the independent variable by putting $x=e^z$, the equation reduces to

$$\{\ D\ (D-1)\ (D-2)-D\ (D-1)+2D-2\ \}\ y=e^{2z}+3e^z$$

or $$(D^3-4D^2+5D-2)\ y=e^{3z}+3e^z$$

or $$(D-1)^2\ (D-2)\ y=e^{3z}+3e^z$$

The $C.F=(c_1+c_2\ z)\ e^z+c_3\ e^{2z}=(c_1+c_2 \log x)\ x+c_3\ x^2.$

$$P.\ I.=\frac{1}{(D-1)^2\,(D-2)}\ (e^{3z}+3e^z),$$

$$=\frac{1}{4}\ e^{3z}-3\ .\ \frac{1}{(D-1)^2}\ e^z,$$

$$=\frac{1}{4}\ e^{3z}-3\ .\ e^z\ .\ \frac{1}{D^2}\ .\ 1=\frac{1}{4}\ e^{3z}-\frac{3}{2}\ e^z\ .\ z^2,$$

$$=\tfrac{1}{4}\ x^3-\tfrac{3}{2}\ x\ (\log x)^2.$$

$\therefore$ The complete integral is

$$y=(c_1+c_2\log x)\ x+c_3\ x^2+\tfrac{1}{4}\ x^3-\tfrac{3}{2}\ x\ (\log x)^2.$$

EXERCISE VI (A)

Solve :

1. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=0.$

2. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=2\log x.$ (*Punjab* 1953, *Agra* 1949)

3. $x^3\dfrac{d^3y}{dx^3}+4x^2\dfrac{d^2y}{dx^2}-2y=0.$

4. $x^2\dfrac{d^3y}{dx^3}-2\dfrac{dy}{dx}=0.$

5. $x^3\,y_3-3x^2\,y_2+6xy_1-6y=(\log x)^2$ (*I. A. S.* 1953)

6. $\dfrac{d^3y}{dx^3}-\dfrac{4}{x}\dfrac{d^2y}{dx^2}+\dfrac{5}{x^2}\dfrac{dy}{dx}-\dfrac{2y}{x^3}=1.$

6·2. *Alternative Method.*

To determine the complementary function. If in the differenal equation

$$x^n\frac{d^ny}{dx^n}+A_1\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+A_n\,y=X, \qquad \ldots(1)$$

we put $y=x^m$, the left hand side of the equation becomes

$$[\,m(m-1)(m-2)\ldots(m-n+1)$$
$$+A_1\,m\,(m-1)\ldots(m-n+2)+\ldots+A_n\,]\,x^m$$

Hence if

$$f\,(m)\equiv m\,(m-1)\ldots(m-n+1)$$
$$+A_1\,m\,(m-1)\ldots(m-n+2)+\ldots+A_n=0 \qquad \ldots(2)$$

the substitution $y=x^m$ makes the first member of (1) vanish. Hence $y=x^m$ is a part of the $C.F.$ of the integral of (1).

Case I. *Distinct unequal roots.* If $m_1, m_2, \ldots\ldots\ldots, m_n$ are n distinct roots of (2), then

$$C.F. = c_1 x^{m_1} + c_2 x^{m_2} + \ldots\ldots + c_n x^{m_n}.$$

Case II. *Equal roots.* Here $f(m)$ in equation (2) is the same as $f(D)$ in equation (2) of the Art. 6·1. Thus corresponding to an integral $y = x^{m_1}$ of (1), there is an integral $y = e^{m_1 z}$ of equation (1) of Art. 6·1. Hence, as has already been seen an integral of (1) can be obtained by putting $z = \log x$ in the integral of (1) Art. 6·1. Therefore if $f(m) = 0$ of (2) has two equal roots say $m_1 = m_2$, the corresponding integral of equation (2) being

$$y = (c_1 + c_2 z)\, e^{m_1 z}$$

the integral of (1) by substituting $z = \log x$, is

$$y = (c_1 + c_2 \log x)\, x^m.$$

In like manner if there are r roots equal $(m_1 = m_2 = \ldots = m_r)$, the corresponding integral of equation (2) is

$$y = [\, c_1 + c_2 \log x + c_3 (\log x)^2 + \ldots\ldots + c_r (\log x)^{r-1}\,]\, x^{m_1}.$$

Case III. *Imaginary roots.* If equation (2) has a pair of imaginary roots, say, $\alpha \pm \beta i$, the corresponding integral of (2) of Art. 6·1 being

$$y = [c_1 \cos(\beta z) + c_2 \sin(\beta z)\,]\, e^{\alpha z}$$

the integral of (1) is

$$y = [\, c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)]\, x^{\alpha}.$$

Ex. 4. *Solve* : $x^3 \dfrac{d^3y}{dx^3} + 6x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} - 4y = 0$ ......(1)

Substituting $y = x^m$ in (1) we get

$$[\, m(m-1)(m-2) + 6m(m-1) + 4m - 4\,]\, x^m = 0$$

or $$(m^3 + 3m^2 - 4)\, x^m = 0$$

or $$(m-1)(m+2)^2\, x^m = 0$$

Substitution of $y = x^m$ will satisfy the equation (1) if

$$(m-1)(m+2)^2 = 0.$$

The roots of this equation are $1, -2, -2$.

Hence the solution is

$$y = c_1 x + (c_2 + c_3 \log x)\, x^{-2}.$$

Ex. 5. *Solve* : $(x^3 D^3 + 3x^2 D^2 + xD + 1)\, y = 0.$ ......(1)

Substituting $y = x^m$ in (1) we get

$$(m^3 + 1)\, x^m = 0\,;$$

Substitution of $y = x^m$ will satisfy the equation (1)

$$m^3 + 1 = 0$$

of which the roots are $-1, \tfrac{1}{2}(1 \pm \sqrt{3}i)$.

Hence the solution is

$$y=c_1\, x^{-1}+\left\{c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right)+c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\} x^{\frac{1}{2}}.$$

EXERCISE VI (B)

Solve :

1. $x^2 \dfrac{d^3y}{dx^3}+x\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}=0.$

2. $x^3 \dfrac{d^3y}{dx^3}-3x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}-8y=0.$

3. $x^2 \dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+5y=0.$ 4. $x^2 \dfrac{d^3y}{dx^3}+3x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=0.$

6·3. Symbolic Notation in θ. We have assumed in Art. 6·1. that the symbol D stands for $\dfrac{d}{dz}$. But as proved before $\dfrac{d}{dz}\equiv x\dfrac{d}{dx}$, hence the expression

$$x^r\frac{d^r}{dx^r}=D\,(D-1)\ldots(D-r+1)$$

may be written as

$$x^r\frac{d^r}{dx^r}=x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\ldots\left(x\frac{d}{dx}-r+1\right).$$

Since D has already been used for $\dfrac{d}{dz}$ in previous chapters, so if we write θ for $x\dfrac{d}{dx}$, the above expression takes the form of

$$x^r\frac{d^r}{dx^r}=\theta\,(\theta-1)\ldots\ldots(\theta-r+1)$$

and the equation

$$x^n\frac{d^ny}{dx^n}+A_1\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+A_n\,y=X$$

is reduced to

$$\{\theta(\theta-1)\ldots(\theta-n+1)+A_1\,\theta(\theta-1)\ldots(\theta-n+2)+\ldots+A_n\}\,y=X.$$

If for the expression in the bracket we write $f(\theta)$, it becomes

$$f(\theta)\, y = X.$$

Here $f(\theta)$ is the same as $f(m)$ of equation (2) of Art. 6·2. Hence the *C.F.* will be obtained in the way as we did in that Article.

Note 1. *It may be noted here that the factors of $f(\theta)$ are commutative. For example*

$$(\theta-\alpha)(\theta-\beta)\, y = \left(x\frac{d}{dx}-\alpha \right)\left(x\frac{dy}{dx}-\beta y \right)$$

$$= x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\beta x\frac{dy}{dx}-\alpha x\frac{dy}{dx}+\alpha\beta y$$

$$= x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}(1-\alpha-\beta)+\alpha\beta y$$

and $$(\theta-\beta)(\theta-\alpha)\, y = \left(x\frac{d}{dx}-\beta \right)\left(x\frac{dy}{dx}-\alpha y \right)$$

$$= x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\alpha x\frac{dy}{dx}-\beta x\frac{dy}{dx}+\beta\alpha y$$

$$= x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}(1-\alpha-\beta)+\alpha\beta y.$$

The above truth can be established by taking even more factors.

Note 2 *The expression $\frac{1}{f(\theta)}X$ may be defined as a function of x which when operated by $f(\theta)$ gives X.*

6·4. To determine the Particular Integral. The determination of the *P. I.*, which is symbolically expressed by $\frac{1}{f(\theta)}X$, may, by the resolution of $\frac{1}{f(\theta)}$ into partial fractions, be reduced to the evaluation of expression of the form

$$\left(\frac{p_1}{\theta-\alpha_1}+\frac{p_2}{\theta-\alpha_2}+\dots\dots\dots+\frac{p_n}{\theta-\alpha_n} \right) X$$

or

$$\frac{p_1}{\theta-\alpha_1}X+\frac{p_2}{\theta-\alpha_2}X+\dots\dots\dots+\frac{p_n}{\theta-\alpha_n}X$$

Thus we have to operate X by operators of the form $\frac{1}{\theta-\alpha}$, p_1, p_2 etc. being constant multipliers.

Let $$\frac{1}{\theta-\alpha}X = y. \qquad \dots(1)$$

We may then write $(\theta-a)\ y=X.$ (2)

Our assumption is legitimate because if we operate both sides of (2) by $\dfrac{1}{\theta-a}$ we get (1).

(2) can be written

$$x\frac{dy}{dx}-a\,y=X, \quad \text{or} \quad \frac{dy}{dx}-\frac{a}{x}\,y=\frac{X}{x}.$$

Integrating we get $\quad ye^{-\alpha\log x}=\int x^{-\alpha-1}\,X\,dx$

$$\therefore \quad y=x^{\alpha}\int x^{-\alpha-1}\,X\,dx.$$

$$\frac{1}{\theta-a}\,X=x^{\alpha}\int x^{-\alpha-1}\,X\,dx.$$

[The students are advised to commit this to memory].

Hence $\quad \dfrac{p_1}{\theta-a_1}\,X=p_1x^{\alpha_1}\displaystyle\int x^{-\alpha_1-1}\,.\,X\,dx$

and therefore the Particular Integral is

$$p_1x^{\alpha_1}\int x^{-\alpha_1-1}\,.\,X\,dx+p_2x^{\alpha_2}\int x^{-\alpha_2-1}\,X\,dx+\ldots$$

$$+p_nx^{\alpha_n}\int x^{-\alpha_n-1}\,X\,dx,$$

Note : *If* $\dfrac{1}{(\theta-\alpha)^r}\,X$ *is one of the partial fractions of* $\dfrac{1}{f(\theta)}\,X$, *the operator* $\dfrac{1}{\theta-\alpha}$ *is to be applied r times to X i. e,*

$$\frac{1}{(\theta-\alpha)^2}=\frac{1}{\theta-\alpha}\,x^{\alpha}\int x^{-\alpha-1}\,.\,X\,dx$$

$$=x^{\alpha}\int x^{-1}\int x^{-\alpha-1}\,X\,dx$$

thus in general

$$\frac{1}{(\theta-\alpha)^r}\,X=x^{\alpha}\int x^{-1}\int x^{-1}\int\ldots\int x^{-\alpha-1}\,X\,dx^r.$$

6·5. Particular Cases. When X is of the form of x^m.

We know that

$$\theta\,x^m=x\frac{d}{dx}\,x^m=x\,mx^{m-1}=mx^m,$$

$$\theta^2 x^m = x\frac{d}{dx}(mx^m) = x.\, m.\, mx^{m-1} = m^2 x^m$$

$$\theta^3 x^m = x\frac{d}{dx}(m^2 x^m) = x.\, m^2.\, mx^{m-1} = m^3 x^m$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

Thus $f(\theta)\, x^m = f(m)\, x^m$. ......(1)

Operating both the sides of (1) by $\dfrac{1}{f(\theta)}\, x^m$, we get

$$\frac{1}{f(\theta)}\{f(\theta)x^m\} = \frac{1}{f(\theta)}\{f(m)x^m\}$$

or $\quad x^m = f(m).\, \dfrac{1}{f(\theta)}\, x^m$, $f(m)$ is a constant

or $$\frac{1}{f(\theta)}\, x^m = \frac{1}{f(m)}\, x^m.$$

Note : If $f(\theta)=0$ has a root $\theta=m$, then $f(m)=0$, hence this method fails.

Let $f(\theta)=(\theta-m)\, F(\theta)$ in this case.

Then the $P.I. = \dfrac{1}{\theta-m}\, \dfrac{1}{F(m)}\, x^m = \dfrac{1}{F(m)} \cdot \dfrac{1}{\theta-m}\, x^m$

$$= \frac{1}{F(m)}\, x^m \int x^{-m-1}.\, x^m\, dx = \frac{x^m \log x}{F(m)}$$

If $f(\theta)=0$ has r roots each equal to m,

$$f(\theta)=(\theta-m)^r F_1(\theta)$$

and therefore the corresponding integral is

$$= \frac{1}{F_1(m)} \cdot \frac{1}{(\theta-m)^r}\, x^m = \frac{x^m (\log x)^r}{r\,!.\, F_1(m)}.$$

Ex. 6. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$. *(Delhi* **1959)**

Since $x\dfrac{d}{dx} = \theta$ and $x^2 \dfrac{d^2}{dx^2} = \theta.(\theta-1)$, the differential equation can be written in the form

$$\{\theta(\theta-1) - 3\theta + 4\}\, y = 2x^2$$

or $$(\theta-2)^2\, y = 2x^2.$$

The complementary function is

$$x^2 (c_1+c_2 \log x).$$

The $$P.\ I.=\frac{1}{(\theta-2)^2}\,2x^2$$

$$=\frac{2}{\theta-2}\left\{ x^2 \int x^2\, x^{-3}\, dx\right\}=\frac{2}{\theta-2}\,(x^2 \log x)$$

$$=2\,x^2 \int x^{-3}\, x^2 \log x\, dx=x^2\ (\log x)^2.$$

Hence the general solution of the given differential equation is

$$y=x^2\ (c_1+c_2 \log x)+(x \log x)^2.$$

Ex. 7. *Solve* :

$$x^3\ \frac{d^3y}{dx^3}+2x^2\ \frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x} \right).$$

(*Delhi* 1961, *Agra* 1952)

Here $f(\theta)$ is $\theta\ (\theta-1)\ (\theta-2)+2\theta\ (\theta-1)+2$, which reduces to

$$(\theta+1)\ (\theta^2-2\theta+2),$$

the roots of which are $-1,\ 1\pm i$.

Hence $C.F.=c_1\, x^{-1}+x\ (c_2 \cos \log x+c_3 \sin \log x)$.

$$P.\ I.=\frac{1}{(\theta+1)\,(\theta^2-2\theta+2)}\cdot 10\left(x+\frac{1}{x}\right)$$

$$=\frac{1}{(\theta+1)(\theta^2-2\theta+2)}\cdot 10x+\frac{1}{(\theta+1)(\theta^2-2\theta+2)}\ 10x^{-1}$$

$$=\frac{10}{2}\ x+\frac{10}{5\ (\theta+1)}\ x^{-1}$$

$$=5x+2x^{-1}\int x^{1-1}.\ x^{-1}\, dx=5x+2x^{-1} \log x.$$

Therefore the solution is

$$y=x\ (c_2 \cos \log x+c_3 \sin \log x+5)+x^{-1}\ (c_1+2 \log x)$$

Ex. 8. *Solve* : $(x^2\, D^2+3xD+1)\ y=\dfrac{1}{(1-x)^2}$.

(*Agra* 1945, '51, '53, '57, *Raj.* '52)

Here $f(\theta)$ is $\theta(\theta-1)+3\theta+1$ which reduces to $(\theta+1)^2$.

Hence $C\ F.=(c_1+c_2 \log x)\ x^{-1}$,

$$P.I=\frac{1}{(\theta+1)^2}\ (1-x)^{-2}$$

$$=\frac{1}{\theta+1}.x^{-1}\int x^{1-1}(1-x)^{-2}\,dx$$

$$=\frac{1}{\theta+1}\,x^{-1}(1-x)^{-1}$$

$$=x^{-1}\int x^{1-1}.\,x^{-1}(1-x)^{-1}\,dx$$

$$=x^{-1}\int\frac{1}{x(1-x)}\,dx=x^{-1}\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx$$

$$=x^{-1}\log\frac{x}{x-1}.$$

Hence the complete solution is

$$y=x^{-1}(c_1+c_2\log x)+x^{-1}\log\frac{x}{x-1}.$$

6·6. Equations which can be reduced to the homogeneous linear form. The equations which are of the form

$$(a+bx)^n\frac{d^ny}{dx^n}+A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots\ldots$$

$$+A_{n-1}(a+bx)\frac{dy}{dx}+A_n\,y=F(x)\qquad\ldots\ldots(1)$$

where $A_1, A_2,\ldots\ldots\ldots,A_n$ are constants, can easily be reduced to the homogeneous linear form with constant coefficients. If we write $z=a+bx$, then

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=b.\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{b}\frac{dy}{dz}\right)=\frac{d}{dz}\left(b\cdot\frac{dy}{dz}\right)\frac{dz}{dx}=b^2\frac{d^2y}{dz^2}$$

$$\ldots\ldots\qquad\ldots\ldots\qquad\ldots\ldots$$

$$\frac{d^ny}{dx^n}=b^n\frac{d^ny}{dz^n}.$$

Substituting these values in (1) it is reduced to

$$z^n\frac{d^ny}{dz^n}+\frac{A_1}{b}z^{n-1}\frac{d^{n-1}y}{dz^{n-1}}+\frac{A_2}{b^2}z^{n-2}\frac{d^{n-2}y}{dz^{n-2}}\ldots\ldots\ldots$$

$$+\frac{A_{n-1}}{b^{n-1}}z\frac{dy}{dz}+\frac{A_n}{b^n}y=\frac{1}{b^n}F\left(\frac{z-a}{b}\right).\qquad\ldots\ldots(2)$$

This is a standard form and can be integrated easily. If $y=F(z)$ be the solution of (1), then $y=F(a+bx)$ is the solution of (1).

Ex. 9. *Solve* : $(2x-1)^3\dfrac{d^3y}{dx^3}+(2x-1)\dfrac{dy}{dx}-2y=0$ ......(1)

Let $2x-1=z$, then $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=2\dfrac{dy}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(2\frac{dy}{dz}\right)=\frac{d}{dz}\left(2\frac{dy}{dz}\right)\frac{dz}{dx}=4\frac{d^2y}{dz^2}$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(4\frac{d^2y}{dz^2}\right)=\frac{d}{dz}\left(4\frac{d^2y}{dz^2}\right)\frac{dz}{dx}=8\frac{d^3y}{dz^3}.$$

Substituting these values in the equation (1), it is reduced to

$$8z^3\frac{d^3y}{dz^3}+2z\frac{dy}{dz}-2y=0 \qquad \text{......(2)}$$

Putting $y=z^m$, we get

$$\{8m(m-1)(m-2)+2m-2\}z^m=0$$

or $$(4m^3-12m^2+9m-1)z^m=0$$

and the roots of the equation are

$$1,\ \tfrac{1}{2}(2+\sqrt{3}) \text{ and } \tfrac{1}{2}(2-\sqrt{3}).$$

Hence the solution of (2) is

$$y=c_1z+c_2z^{1+\frac{1}{2}\sqrt{3}}+c_3z^{1-\frac{1}{2}\sqrt{3}}$$

and therefore the solution of (1) is

$$y=c_1(2x-1)+c_2(2x-1)^{1+\frac{1}{2}\sqrt{3}}+c_3(2x-1)^{1-\frac{1}{2}\sqrt{3}}$$

$$=(2x-1)\left\{c_1+c_2(2x-1)^{\frac{1}{2}\sqrt{3}}+c_3(2x-1)^{-\frac{1}{2}\sqrt{3}}\right\}$$

Ex. 10. *Solve* :

$$(x+a)^2\frac{d^2y}{dx^2}-4(x+a)\frac{dy}{dx}+6y=x. \quad (\textit{Agra } 1956, 62)$$

Let $x+a=z$, then $\dfrac{dy}{dx}=\dfrac{dy}{dz}$ and $\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dz^2}$.

Substituting these values in the given equation it is reduced to

$$z^2\frac{d^2y}{dz^2}-4z\frac{dy}{dz}+6y=z-a.$$

Or in a symbolic form it is reduced to

$$\{\theta(\theta-1)-4\theta+6\}y=z-a$$

i.e. $$(\theta-2)(\theta-3)y=z-a.$$

The $C.F.=c_1 z^2+c_2 z^3=c_1 (x+a)^2+c_2 (x+a)^3$

$$P.I.=\frac{1}{(\theta-2)(\theta-3)}(z-a)$$

$$=\left(\frac{1}{\theta-3}-\frac{1}{\theta-2}\right)(z-a)$$

$$=z^3\int z^{-4}(z-a)\,dz-z^2\int z^{-3}(z-a)\,dz$$

$$=\tfrac{1}{2}z-\tfrac{1}{6}a=\tfrac{1}{2}(x+a)-\tfrac{1}{6}a=\tfrac{1}{6}(3x+2a).$$

Hence complete solution is

$$y=c_1 (x+a)^2+c_2 (x+a)^3+\tfrac{1}{6}(3x+2a).$$

Ex. 11. *Solve* :

$$16(x+1)^4\frac{d^4y}{dx^4}+96(x+1)^3\frac{d^3y}{dx^3}+104(x+1)^2\frac{d^2y}{dx^2}+8(x+1)\frac{dy}{dx}+y=x^2+4x+3.$$

Substituting $x+1=z$, the given equation is reduced to

$$16z^4\frac{d^4y}{dz^4}+96z^3\frac{d^3y}{dz^3}+104z^2\frac{d^2y}{dz^2}+8z\frac{dy}{dz}+y=(z+1)^2-1=z^2+2z.$$

In symbolic form it is transformed to

$$(16\theta^4-8\theta^2+1)\,y=z^2+2z$$

or

$$(2\theta-1)^2(2\theta+1)^2\,y=z^2+2z.$$

Hence the $C.F.=(c_1+c_2\log z)\,z^{\frac{1}{2}}+(c_3+c_4\log z)\,z^{-\frac{1}{2}}.$

$$=\{c_1+c_2\log(x+1)\}(x+1)^{\frac{1}{2}}$$

$$+\{c_3+c_4\log(x+1)\}(x+1)^{-\frac{1}{2}}.$$

$$P.I.=\frac{1}{(2\theta-1)^2(2\theta+1)^2}(z^2+2z)$$

$$=\frac{z^2}{9\times 25}+\frac{2z}{9}=\frac{z^2+50z}{225}=\frac{x^2+52x+51}{225}$$

Hence $C.F.$ and $P.I.$ together give the complete solution.

Ex. 12. *Solve* $(1+x)^2 y_2+(1+x)\,y_1+y=4\cos\log(1+x)$

(*Agra* 1950)

Putting $1+x=e^z$ and denoting $\frac{d}{dz}$ by the symbol D, the differential equation is reduced to

$$D(D-1)y+Dy+y=4\cos z.$$

or
$$(D^2+1)y=4\cos z.$$

The auxiliary equation is

$$m^2+1=0$$

$\therefore$ $C.F.=c_1\cos z+c_2\sin z=c_1\cos\log(1+x)+c_2\sin\log(1+x)$

$$P.I.=4\frac{1}{D^2+1}\cos z=4.\tfrac{1}{2}z\sin z$$

$$=2\log(1+x)\sin\log(1+x).$$

$\therefore$ The complete solution is

$y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$

Ex. 13. *Solve* :

$$2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx},$$

after making it homogeneous by the substitution $y=z^2$.

Since $y=z^2$, therefore $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=2z\dfrac{dz}{dx}$.

$$\frac{d^2y}{dx^2}=2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}.$$

Substituting these values in the given equation it is reduced to

$$x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$$

which is a homogeneous linear equation. In symbolic notation it is

$$(\theta^2-2\theta+1)z=0.$$

Hence its solution is

$$z=(c_1+c_2\log x)x$$

or
$$y=(c_1+c_2\log x)^2x^2.$$

Ex. 14. *Solve* :

$$[x^2D^2-(2m-1)xD+(m^2+n^2)]y=n^2x^m\log x.$$

Substituting $x=e^z$ *i e.* $z=\log x$, the equation is reduced to

$$\frac{d^2y}{dz^2}-(2m-1)\frac{dy}{dz}+(m^2+n^2)y=n^2e^{mz}.z \qquad \ldots\ldots(1)$$

If $D'=\dfrac{d}{dz}$, then the equation (1) is

$$[D'(D'-1)-(2m-1)D'+m^2+n^2]y=n^2e^{mz}.z$$

i.e. $(D'^2-2mD'+m^2+n^2)y=n^2e^{mz}.z.$

Roots of the coefficient of y are $m\pm ni$.

Hence $C.\ F.=x^m\ [\ c_1 \cos\ (n \log x)+c_2 \sin\ (n \log x)\ ]$

and $$P.\ I.=\frac{1}{D'^2-2mD'+m^2+n^2}\ n^2\ e^{mz}.\ z$$

$$=e^{mz}\ .\ \frac{1}{(D'+m)^2-2m\ (D'+m)+m^2+n^2}\ .\ n^2\ z$$

$$=e^{mz}\ \frac{1}{D'^2+n^2}\ n^2\ z$$

$$=n^2\ e^{mz}\ .\ \frac{1}{n^2}\left(1-\frac{D'^2}{n^2}\right)z$$

$$=e^{mz}\ .\ z=x^m \log x.$$

Hence solution is

$$y=x^m\ [\ c_1 \cos\ (n \log x)+c_2 \sin\ (n \log x)\ ]+x^m \log x.$$

Ex. 15. *Solve* :

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\log x.\ \sin\ (\log x)+1}{x}.$$ (*Agra* 1939, 1962)

On changing the independent variable by putting $x=e^z$, this equation becomes

$$(D^2-4D+1)\ y=e^{-z}\ .\ z \sin z+e^{-z}$$

The complementary function

$$=c_1\ e^{(2+\sqrt{3})z}+c_2 e^{(2-\sqrt{3})z}=c_1\ x^{2+\sqrt{3}}+c_2\ x^{2-\sqrt{3}}.$$

$$=x^2\ (c_1\ x^{\sqrt{3}}+c_2\ x^{-\sqrt{3}})$$

and Particular Integral

$$=\frac{1}{D^2-4D+1}\ e^{-z}\ .\ z \sin z+\frac{1}{D^2-4D+1}\ e^{-z}$$

$$=e^{-z}\ .\ \frac{1}{D^2-6D+6}\ z \sin z+\frac{1}{6}\ e^{-z}.$$

$$=e^{-z}\ \left\{z\ .\ \frac{1}{D^2-6D+6} \sin z-\frac{2D-6}{(D^2-6D+6)^2} \sin z\ \right\}+\frac{1}{6}\ e^{-z}$$

$$=e^{-z}\ \left\{z\ .\ \frac{1}{5-6D} \sin z-\frac{2D-6}{(5-6D)^2} \sin z\ \right\}+\frac{1}{6}\ e^{-z}$$

$$=e^{-z}\ \left\{z\ .\ \frac{5+6D}{61} \sin z+\frac{2D-6}{11+60D} \sin z\ \right\}+\frac{1}{6}\ e^{-z}$$

$$=e^{-z}\left\{z\frac{(5\sin z+6\cos z)}{61}+\frac{54\sin z+382\cos z}{3721}\right\}+\frac{1}{6}e^{-z}$$

$$=x^{-1}\{\log x\,(5\sin\log x+6\cos\log x)/61$$
$$+(54\sin\log x+382\cos\log x)/3721+\tfrac{1}{6}\}$$

C. F. and *P. I.* together give the complete solution.

EXERCISE VI (C)

Solve :

1. $x^2\dfrac{d^2y}{dx^2}+y=3x^2.$

2. $x^2\dfrac{d^2y}{dx^2}+7x\dfrac{dy}{dx}+5y=x^5.$ (*Nagpur* 1948)

3. $x^2\dfrac{dy}{dx}+5x\dfrac{dy}{dx}+4y=x^4.$

4. $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=x^4$ (*I. A. S.* 1955)

5. $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4.$ (*Raj.* 1958, *Agra* 1958, *Alld.* 1953)

6. $(x^2D^2+xD-1)\,y=x^m.$ (*Bombay* 1936)

7. $(x^2D^2-3xD+4)\,y=x^m.$

8. $(x^2D^2+2xD)\,y=\log x.$ (*Nagpur* 1961)

9. $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=e^x.$ (*L. I. C.* 1960 ; *Agra* 1946)

10. $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}-3y=X.$

11. $x^2\dfrac{d^3y}{dx^3}+3x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=x.$

12. $x^4\dfrac{d^4y}{dx^4}+6x^3\dfrac{d^3y}{dx^3}+9x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=4x.$

13. $x^3\dfrac{d^3y}{dx^3}+6x^2\dfrac{d^2y}{dx^2}+8x\dfrac{dy}{dx}+2y=x^2+3x-4.$ (*Nagpur* 1953)

14. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-20y=(x+1)^2.$

15. $x^3 \frac{d^3y}{dx^3} - 3x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} - 8y = x^2 + \frac{1}{x^2}$ (*Nagpur* 1958)

16. $x^4 \frac{d^4y}{dx^4} + 2x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = x + \log x.$ (*Mysore* 1949)

17. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x.$ (*Punjab* 1952, *Lucknow* 1948)

18. $(x^2 D^2 - 3x D + 5) y = x^2 \sin (\log x).$ (*Osmania* 1960)

19. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = x \log x.$

20. $(x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3x D + 1) y = (1 + \log x)^2.$

21. $(5+2x)^2 \frac{d^2y}{dx^2} - 6 (5+2x) \frac{dy}{dx} + 8y = 0.$

22. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} = (2x+3)(2x+4).$

CHAPTER VII

EXACT DIFFERENTIAL EQUATIONS AND CERTAIN PARTICULAR FORMS OF EQUATIONS

7·1. Definition. *A differential equation of the form*

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots\ldots\ldots, \frac{dy}{dx}, y\right)=\phi(x) \qquad \ldots\ldots(1)$$

is said to be exact when it can be obtained by differentiating directly an equation of the next lower order of the form

$$f\left(\frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots\ldots, \frac{dy}{dx}, y\right)=\int \phi(x)\, dx+c \qquad \ldots\ldots(2)$$

The equation (2) is said to be a first integral of (1)

7·2. Condition of exactness of a linear differential Equation.

Consider the differential equation

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots\ldots+P_ny=\phi(x) \qquad \ldots\ldots(1)$$

where $P_0, P_1, \ldots\ldots, P_n$ are functions of x. Now we are to establish a certain relation between P's so that the equation may be exact. Let the successive derivatives be denoted by dashes *i.e.*

$$P', P'', \ldots\ldots\ldots P^{(n)}.$$

Now we have on direct integration

$$\int P_n y\,dx=\int P_n y\,dx$$

$$\int P_{n-1}\frac{dy}{dx}dx=P_{n-1}y-\int P'_{n-1}y\,dx$$

$$\int P_{n-2}\frac{d^2y}{dx^2}dx=P_{n-2}\frac{dy}{dx}-\int P'_{n-2}\frac{dy}{dx}dx$$

$$=P_{n-2}\frac{dy}{dx}-P'_{n-2}y+\int P''_{n-2}y\,dx$$

$$\int P_{n-3}\frac{d^3y}{dx^3}dx=P_{n-3}\frac{d^2y}{dx^2}-P'_{n-3}\frac{dy}{dx}+P''_{n-3}y-\int P'''_{n-3}y\,dx$$

$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots\ldots$$

Thus term by term integration of the differential equation gives

$$\int (P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \ldots\ldots)\, y dx$$

$$+ (P_{n-1} - P'_{n-2} + P''_{n-3} - \ldots\ldots)\, y + (P_{n-2} - P'_{n-3} + \ldots\ldots) \frac{dy}{dx}$$

$$+ (P_{n-3} - P'_{n-4} + \ldots) \frac{d^2y}{dx^2} + \ldots + P_0 \frac{d^{n-1}y}{dx^{n-1}} = \int \phi(x)\, dx + c$$

Now the condition of being exact is evidently that there shall be no term remaining which involves an integral of y, and therefore the condition is

$$\mathbf{P}_n - \mathbf{P}'_{n-1} + \mathbf{P}''_{n-2} - \mathbf{P}'''_{n-3} + \ldots\ldots + (-1)^{n-1} \mathbf{P}_0^{n-1} = 0 \quad \ldots\ldots(2)$$

When this condition is satisfied, the first integral is

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P'_0) \frac{d^{n-2}y}{dx^{n-2}} + (P_2 - P'_1 + P''_0) \frac{d^{n-3}y}{dx^{n-3}}$$

$$+ \ldots + \{ P_{n-1} - P'_{n-2} + \ldots + (-1)^n P_0^{n-1} \}\, y = \int \phi(x)\, dx + c$$

Aliter.* If Equation (1) is an exact, let its first integral be

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + \phi_1 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + \phi_{n-1}(x)\, y = \int \phi\, dx + C$$

where $\phi_1, \phi_2, \ldots\ldots, \phi_{n-1}$ are functions of x.

Differentiating it with respect to x and comparing it with (1), we get

$$P_1 = P'_0 + \phi_1,\ P_2 = \phi'_1 + \phi_2,\ P_3 = \phi'_2 + \phi_3,$$
$$P_{n-1} = \phi'_{n-2}, + \phi_{n-1},\ P_n = \phi'_{n-1}$$

where dashes denote the differentiation with respect to x.

Out of these n equations, from first $(n-1)$ we get

$$\phi_1 = P_1 - P'_0,\ \phi_2 = P_2 - P'_1 - P''_0$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$\phi_{n-1} = P_{n-1} - P'_{n-2} + P''_{n-3} - \ldots\ldots + (n-1)^{n-1} P_0^{n-1}$$

Substituting the value of the derivatives of ϕ_{n-1} from the last equation in $P_n = \phi'_{n-1}$ we get the required condition.

Ex. 1. *Solve* : $(1+x+x^2) \dfrac{d^3y}{dx^3} + (3+6x) \dfrac{d^2y}{dx^2} + 6 \dfrac{dy}{dx} = 0$

(*Agra* 1949, '56, *Jaipur* 1950)

Here $P_0 = 1+x+x^2$, $P_1 = 3+6x$, $P_2 = 6$, $P_3 = 0$. The condition of being exact is satisfied. Hence the equation is exact and on direct integration we get

$$(1+x+x^2) \frac{d^2y}{dx^2} + (2+4x) \frac{dy}{dx} + 2y = c_1$$

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*See *The Maths. Seminar* March 1959 Vol. I page 40.

The condition of being exact is again satisfied, hence it is also exact and integral is

$$(1+x+x^2)\frac{dy}{dx}+(1+2x)\,y=c_1\,x+c_2.$$

The condition of being exact is again satisfied, hence it is also exact and its integral is

$$(1+x+x^2)\,y=c_1\,\frac{x^2}{2}+c_2\,x+c_3.$$

Ex. 2. *Solve* :

$$(x^3-x)\,\frac{d^3y}{dx^3}+(8x^2-3)\,\frac{d^2y}{dx^2}+14x\,\frac{dy}{dx}+4y=\frac{2}{x^3}.$$

(*Agra* 1946, *Rajasthan* 1949, 1959)

Here $P_0=x^3-x,\ P_1=8x^2-3,\ P_2=14x,\ P_3=4.$ Writing the condition of being exact $4-14+16-6=0.$

Hence the equation is exact. Integration gives

$$(x^3-x)\,\frac{d^2y}{dx^2}+(5x^2-2)\frac{dy}{dx}+4xy=-\frac{1}{x^2}+c.$$

Again, on applying the condition of being exact

$$4x-10x+6x=0\ ;$$

hence it is also exact and its integral is

$$(x^3-x)\,\frac{dy}{dx}+(2x^2-1)\,y=\frac{1}{x}+c_1\,x+c_2\ ;$$

This equation is not exact, for $2x^2-1-(3x^2-1)$ is not equal to zero ; but it is a linear equation of the first order *i.e.*

$$\frac{dy}{dx}+\frac{2x^2-1}{x^3-x}\,y=\frac{1}{x(x^3-x)}+\frac{c_1}{x^2-1}+\frac{c_2}{x^3-x}$$

which on integration gives

$$xy\,\sqrt{(x^2-1)}=\sec^{-1}x+c_1\,\sqrt{(x^2-1)}+c_2\log\left[x+\sqrt{(x^2-1)}\,\right]+c_3.$$

Ex. 3. *Solve* :

$$\frac{d^3y}{dx^3}+\cos x\,\frac{d^2y}{dx^2}-2\sin x\frac{dy}{dx}-y\cos x=\sin 2x$$

(*Allahabad* 1953, *Agra* 1957)

It satisfies the condition of being exact *i.e.*

$-\cos x+2\cos x-\cos x=0.$ Hence it is exact.

Integrating

$$\frac{d^2y}{dx^2}+\cos x\,\frac{dy}{dx}-y\sin x=-\tfrac{1}{2}\cos 2x+c_1$$

The equation is still exact. Hence integrating again

$$\frac{dy}{dx}+y\cos x=-\tfrac{1}{4}\sin 2x+c_1x+c_2$$

This equation is linear. Hence integrating

$$y\,e^{\sin x}=\int -\tfrac{1}{4}\sin 2x\,.\,\sin x\,dx+\int e^{\sin x}(c_1x+c_2)\,dx+c_3$$

$$=-\tfrac{1}{2}\int \sin x\cos x\,e^{\sin x}\,dx+\int e^{\sin x}(c_1x+c_2)\,dx+c_3$$

Now $\displaystyle\int \sin x\cos x\,e^{\sin x}\,dx=\sin x\,e^{\sin x}-\int \cos x\,e^{\sin x}\,dx$

$$=\sin x\,e^{x}-e^{\sin x}$$

$$\therefore\quad y\,e^{\sin x}=-\tfrac{1}{2}(\sin x-1)\,e^{\sin x}=\int e^{\sin x}(c_1x+c_2)\,dx+c_3$$

or $\displaystyle y=-\tfrac{1}{2}(\sin x-1)+e^{-\sin x}\int e^{\sin x}(c_1x+c_2)\,dx+c_3\,e^{-\sin x}$

7·3. Integrating Factors. If may be noticed here that in many cases $P_0, P_1, \ldots\ldots, P_n$ are either of the form ax^s or sums of expressions of this form ; and $x^s\dfrac{d^ry}{dx^r}$ is an exact differential coefficient if $s < r$ because on integrating it by parts we get

$$x^s\frac{d^{r-1}y}{dx^{r-1}}-s\,x^{s-1}\frac{d^{r-2}y}{dx^{r-2}}+s\,(s-1)\,x^{s-2}\frac{d^{r-3}y}{dx^{r-3}}+\ldots\ldots$$

$$+(-1)^s\,s\,!\,.\,\frac{d^{r-s-1}y}{dx^{r-s-1}}.$$

If $r=s+1$, the last term is $(-1)^s\,s\,!\,y$.

Now if s is negative, fractional or an integer equal to or greater than r, the term $x^s\dfrac{d^ry}{dx^r}$ will not be an exact differential coefficient because the condition of integrability contains r^{th} derivative only of the coefficient of $\dfrac{d^ry}{dx^r}$ and in that case a term of the type x^{s-r} will occur in the condition which for different terms may have different values and condition therefore may not be satisfied.

In such cases, however, if we multiply all the terms by x^m and then applying the condition of integrability we may find a particular value of m to satisfy the condition.

Ex. 4. *Solve* : $\sqrt{x}\,\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+3y=x.$ (*Jaipur* 1951)

Here the condition of being exact is not satisfied. However we can make it exact by multiplying it by a factor x^m for a particular value of m.

Multiplying by x^m, we get

$$x^{m+1/2}\,\frac{d^2y}{dx^2}+2x^{m+1}\,\frac{dy}{dx}+3x^m\,y=x^{m+1}$$

Now the condition of being exact will be satisfied if

$$3x^m-2\,(m+1)\,x^m+(m+\tfrac{1}{2})\,(m-\tfrac{1}{2})\,x^{m-3/2}=0$$

or $$-2x^m\,(m-\tfrac{1}{2})+(m+\tfrac{1}{2})\,(m-\tfrac{1}{2})\,x^{m-3/2}=0$$

which is satisfied if $m=\frac{1}{2}$.

Now multiplying all the terms of the original equation by $x^{1/2}$, we get

$$x\,\frac{d^2y}{dx^2}+2x^{3/2}\frac{dy}{dx}+3x^{1/2}\,y=x^{3/2}$$

which is exact. Hence integrating

$$x\,\frac{dy}{dx}+(2x^{3/2}-1)\,y=\frac{2}{5}\,x^{5/2}+c_1$$

or $$\frac{dy}{dx}+\left(2x^{1/2}-\frac{1}{x}\right)y=\frac{2}{5}\,x^{3/2}+\frac{c_1}{x}$$

which is a linear equation of the first order.

The integrating factor$=e^{\int (2x^{\frac{1}{2}}-x^{-1})\,dx}=e^{\frac{4}{3}x^{3/2}}x^{-1}.$

Hence integrating

$$y\,e^{\frac{4}{3}x^{3/2}}.\,x^{-1}=\int\frac{2}{5}\,x^{1/2}\,e^{\frac{4}{3}x^{3/2}}dx+\int c_1\,e^{\frac{4}{3}x^{3/2}}\,x^{-2}\,dx+c_2$$

$$=\frac{1}{5}e^{\frac{4}{3}x^{3/2}}+c_1\int e^{\frac{4}{3}x^{3/2}}.\,x^{-2}\,dx+c_2.$$

Ex. 5. *Solve* : $2x^2\,(x+1)\,\dfrac{d^2y}{dx^2}+x(7x+3)\,\dfrac{dy}{dx}-3y=x^2.$

Here for some terms $s-r=1$ and for some $s-r=0$. Hence multiplying it by a factor x^m, it becomes

$$2x^{m+2}(x+1)\frac{d^2y}{dx^2}+x^{m+1}(7x+3)\frac{dy}{dx}-x^m y=x^{m+2}.$$

Applying the condition of integrability

$$-3x^m-7(m+2)x^{m+1}-3(m+1)x^m$$
$$+2(m+3)(m+2)x^{m+1}+(m+2)(m+1)x^m=0$$

which reduces to

$$(m+2)(2m-1)x^{m+1}+(m+2)(2m-1)x^m=0$$

which is satisfied if $m=-2$ or if $m=\frac{1}{2}$.

Thus multiplying first by x^{-2}, we have the exact equation

$$2(x+1)\frac{d^2y}{dx^2}+\left(7+\frac{3}{x}\right)\frac{dy}{dx}-\frac{3}{x^2}y=1$$

whose integral is

$$2(x+1)\frac{dy}{dx}+\left(5+\frac{3}{x}\right)y=x+c_1 \qquad ...(1)$$

Again multiplying by $\sqrt{x}$, we have the exact equation

$$2(x^{7/2}+x^{5/2})\frac{d^2y}{dx^2}+(7x^{5/2}+3x^{3/2})\frac{dy}{dx}-3x^{1/2}y=x^{5/2}$$

whose integral is

$$2x^{5/2}(x+1)\frac{dy}{dx}-2x^{3/2}y=\frac{2}{7}x^{7/2}+c_2 \qquad ...(2)$$

Now (1) and (2) are both linear and can easily be integrated.

If we eliminate $\frac{dy}{dx}$ from (1) and (2), the integral comes as

$$5(x+1)y=c_1x+c_2x^{-3/2}+\tfrac{5}{7}x^2.$$

Ex. 6. *Solve* :

$$2x^2\cos y\frac{d^2y}{dx^2}-2x^2\sin y\left(\frac{dy}{dx}\right)^2+x\cos y\frac{dy}{dx}-\sin y=\log x.$$

(*Jaipur* 1958)

Substituting $x=e^z$, the equation is reduced to

$$2\cos y\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)-2\left(\frac{dy}{dz}\right)^2\sin y+\cos y\left(\frac{dy}{dz}\right)-\sin y=z.$$

or $$\left\{2\cos y\left(\frac{d^2y}{dz^2}\right)-2\sin y\left(\frac{dy}{dz}\right)^2\right\}-\left\{\cos y\frac{dy}{dz}+\sin y\right\}=z$$

or $$\left\{e^{-z}2\cos y\left(\frac{d^2y}{dz^2}\right)-2\sin y.\,e^{-z}\left(\frac{dy}{dz}\right)^2\right\}$$
$$-\left\{e^{-z}\cos y\frac{dy}{dz}+e^{-z}\sin y\right\}=z\,e^{-z}$$

or $$\left\{e^{-z}.\,2\cos y\frac{d^2y}{dz^2}-2\sin y.\,e^{-z}\left(\frac{dy}{dz}\right)^2-2e^{-z}\cos y\frac{dy}{dz}\right\}$$
$$+\left\{e^{-z}\cos y\frac{dy}{dz}-e^{-z}\sin y\right\}=z\,e^{-z}$$

or $$\frac{d}{dz}\left\{e^{-z}.\,2\cos y\left(\frac{dy}{dz}\right)\right\}+\frac{d}{dz}(e^{-z}\sin y)=z\,e^{-z}$$

Integrating, $e^{-z}.\,2\cos y\,\dfrac{dy}{dz}+\sin y\,e^{-z}=-z\,e^{-z}-e^{-z}+c_1$

or $$2\cos y\frac{dy}{dz}+\sin y=-z-1+c_1\,e^{z}.$$

Again substituting $\sin y=t$, and therefore $\cos y\,dy=dt$,

$$\frac{dt}{dz}+\tfrac{1}{2}\,t=-\tfrac{1}{2}\,z-\tfrac{1}{2}+\tfrac{1}{2}\,c_1\,e^{z}$$

The equation is linear and integrating factor is $e^{z/2}$.

$\therefore\quad t\,e^{z/2}=\int e^{z/2}.\,\tfrac{1}{2}\,z\,dz-\tfrac{1}{2}\int e^{z/2}\,dz+\tfrac{1}{2}\,c_1\int e^{3z/2}\,dz$

$\therefore\quad \sin y.\,e^{z/2}=z\,e^{z/2}-4\,e^{z/2}.+\tfrac{1}{3}\,c_1\,e^{3z/2}+c_2$

or $\quad \sin y=z-4\,e^{z}+c_2\,e^{-z/2}=\log x-4+c_3\,x+c_2/\sqrt{x}.$

7·4. Exactness of non-linear equations.

For equations which are not linear there is no simple test for exactness. The method of integrating non-linear equations which are exact will be as follows. It may be noted that this method can as well be applied in integrating exact equations which are linear.

Let us take the differential equation of n^{th} order. As the equation is derived by differentiation merely it will contain $\dfrac{d^ny}{dx^n}$ in the first degree. If it is not so, the equation is not exact. Suppose the equation is exact.

Writing it in the form $u=0$ integrate $u\,dx$ assuming that $\dfrac{d^{n-1}y}{dx^{n-1}}$ were the only variable in the differential equation and $\dfrac{d^ny}{dx^n}$ its differential coefficient.

Let the result be denoted by u_1 Then $u\,dx-du_1$ will contain differential coefficients upto $(n-1)^{th}$ degree at the utmost.

It may be noted that in finding du_1 the restriction that $\frac{d^{n-1}y}{dx^{n-1}}$ is the only variable is removed.

Repeat the same process often as necessary. We shall ultimately get $udx-du_1-du_2-\ldots\ldots=0$.

Therefore a first integral is $u_1+u_2+\ldots=c$. We shall illustrate the process by solving examples.

Ex. 7. *Solve :* $x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2-3y^2=0$

(*Agra* 1948, '54, '58, *Jaipur* 1948, '50, '58)

We have $x^2y\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}-2y^2=0$

$x^2y\frac{d^2y}{dx^2}$ will arise from differentiation of $x^2y\frac{dy}{dx}$

$\therefore$ Put $u_1=x^2y\frac{dy}{dx}$

Now $du_1=\left[2xy\frac{dy}{dx}dx+x^2\left(\frac{dy}{dx}\right)^2+x^2y\frac{d^2y}{dx^2}\right]dx$

$$udx-du_1=\left(-4xy\frac{dy}{dx}-2y^2\right)dx.$$

$$u_2=-2xy^2,\quad\therefore\quad du_2=\left(-4xy\frac{dy}{dx}-2y^2\right)dx$$

$$u\,dx-du_1-du_2=0$$

$\therefore\quad x^2y\frac{dy}{dx}-2xy^2=C$

Aliter. We may write the equation as

$$\left[x^2y\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}\right]-\left[4xy\frac{dy}{dx}+2y^2\right]=0$$

Integrating terms in the first bracket we get $x^2y\frac{dy}{dx}$

Integrating the terms in the 2nd bracket we get $2y^2x$.

$$\therefore\quad x^2y\frac{dy}{dx}-2y^2x=c.\quad\text{or}\quad y\frac{dy}{dx}-\frac{2}{x}y^2=\frac{c}{x^2}$$

Putting $y^2=v$ and therefore $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$, the eq^n is reduced to

$$\frac{1}{2}\frac{dv}{dx}-\frac{2v}{x}=\frac{c}{x^2} \text{ or } \frac{dv}{dx}-\frac{4v}{x}=\frac{c_1}{x^2} \text{ where } 2c=c_1$$

This equation is linear

$$\therefore\quad v\,x^{-4}=-\frac{1}{5}\frac{c_1}{x^5}+c_2$$

$$\therefore\quad y^2\,x+a_1=a_2\,x^5.$$

Ex. 8. *Show that the following equation is exact and find its first integral.*

$$y+3x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^3+\left(x^2+2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0. \qquad (Agra\ '37)$$

Let us assume that the given equation is exact. Integrating the equation on the understanding that $\dfrac{dy}{dx}$ is the only variable *i. e.* finding the quantity which when differentiated will give

$$\left(x^2+2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2} \text{ we get } u_1=x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2$$

$$\therefore\quad du_1=\left[x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^3+2y^2\frac{dy}{dx}\frac{d^2y}{ax^2}\right]dx$$

$$\therefore\quad u\,dx-du_1=\left[y+x\frac{dy}{dx}\right]dx$$

$$\therefore\quad u_2=xy,\ du_2=\left[x\frac{dy}{dx}+y\right]dx.$$

$$\therefore\quad u\,dx-du_1-du_2=0.$$

Hence our assumption that the equation is exact is legitimate and the first integral is $u_1+u_2=c$,

$$i.\,e.\qquad y^2\left(\frac{dy}{dx}\right)^2+x^2\frac{dy}{dx}+xy=c$$

Aliter. We may also get the integral by grouping the terms as follows

$$\left[x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}\right]+\left[2y^2\frac{dy}{dx}\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^3\right]+\left[x\frac{dy}{dx}+y\right]=0$$

Therefore the first integral is

$$x^2 \frac{dy}{dx}+y^2 \left(\frac{dy}{dx}\right)^2+xy=c.$$

Ex. 9. *Show that the equation*

$$\left(y^2+2x^2 \frac{dy}{dx} \right) \frac{d^2y}{dx^2}+2\,(y+x) \left(\frac{dy}{dx}\right)^2+x \frac{dy}{dx}+y=0$$

is exact and find its first integral.

Writing it in the form

$$\left[y^2 \frac{d^2y}{dx^2}+2y \left(\frac{dy}{dx}\right)^2\right]+\left[2x^2 \frac{dy}{dx} \frac{d^2y}{dx^2}+2x \left(\frac{dy}{dx}\right)^2\right]$$

$$+\left[x \frac{dy}{dx}+y \right]=0,$$

we see that on direct integration the first integral is

$$y^2 \frac{dy}{dx}+x^2 \left(\frac{dy}{dx}\right)^2+xy=c.$$

EXERCISE VII (A)

Solve :

1. $x \dfrac{d^2y}{dx^2}+2x \dfrac{dy}{dx}+2y=0$

2. $y_2+e^x (y_1+y)=e^x$

3. $(1+x^2) \dfrac{d^2y}{dx^2}+3x \dfrac{dy}{dx}+y=0$ (*Agra* 1953)

4. $x^3 \dfrac{d^3y}{dx^3}+6x^2 \dfrac{d^2y}{dx^2}+8x \dfrac{dy}{dx}+2y=x^2+3x-4$ (*Nagpur* 1953)

5. $x \dfrac{d^3y}{dx^3}+(x^2-3) \dfrac{d^2y}{dx^2}+4x \dfrac{dy}{dx}+2y=0$ (*Allahabad* 1952)

6. $\dfrac{d^2y}{dx^2}+2e^x \dfrac{dy}{dx}+2e^x y=x^2.$

7. $(x^2-x) \dfrac{d^2y}{dx^2}+2\,(2x+1) \dfrac{dy}{dx}+2y=0.$

8. $(x^2-x) \dfrac{d^2y}{dx^2}-2\,(x-1) \dfrac{dy}{dx}-4y=0.$

9. $(1-x^2) \dfrac{d^2y}{dx^2}-x \dfrac{dy}{dx}+y=2x.$

10. $(2x^2+3x)\dfrac{d^2y}{dx^2}+(6x+3)\dfrac{dy}{dx}+2y=(x+1)\,e^x.$

11. $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2+y\dfrac{dy}{dx}=0.$

12. $(ax-bx^2)\dfrac{d^2y}{dx^2}+2a\dfrac{dy}{dx}+2by=x$ (*Rajasthan* 1954)

13. $\sin x\dfrac{d^2y}{dx^2}-\cos x\dfrac{dy}{dx}+2y\sin x=0.$ (*Agra* 1954, 1959)

Find the first integrals of :

14. $x^2\dfrac{d^3y}{dx^3}+4x\dfrac{d^2y}{dx^2}+(x^2+2)\dfrac{dy}{dx}+3xy=2.$

[**Hint.** Multiply all the terms by x].

15. $x^5\dfrac{d^6y}{dx^6}+x^4\dfrac{d^5y}{dx^5}+x\dfrac{dy}{dx}+y=\log x$ (*Agra* 1932)

16. $x^3\dfrac{d^3y}{dx^3}+4x^2\dfrac{d^2y}{dx^2}+x(x^2+2)\dfrac{dy}{dx}+3x^2y=2x.$

17. $x^5\dfrac{d^2y}{dx^2}+3x^3\dfrac{dy}{dx}+(3-6x)\,x^2\,y=x^4+2x-5.$

18. Find three independent integrals of

$$\frac{d^3y}{dx^3}=F(x).$$

19. Shew that $y^2+(2xy-1)\dfrac{dy}{dx}+x\dfrac{d^2y}{dx^2}+x^2\dfrac{d^3y}{dx^3}=0$ is an exact differential equation, and deduce a first integral.

7·5. An equation in the form $\dfrac{d^ny}{dx^n}=f(x)$.

The equation can be directly integrated. The first integration gives $\dfrac{d^{n-1}y}{dx^{n-1}}=\int f(x)\,dx+a_1$, where a_1 is an arbitrary constant.

The second gives

$$\frac{d^{n-2}y}{dx^{n-2}}=\iint f(x)\,(dx)^2+a_1x+a_2$$

where a_2 is an arbitrary constant.

Integrating n times we get

$$y=\int\int\ldots\ldots\int f(x)\,(dx)^n+c_1 x^{n-1}+c_2 x^{n-2}+\ldots+c_n.$$

where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.

Ex. 10. *Solve :* $\dfrac{d^3y}{dx^3}=x\,e^x$ (*Allahabad* **1942**)

Integrating, $\dfrac{d^2y}{dx^2}=x\,e^x-e^x+c_1$

Integrating again, $\dfrac{dy}{dx}=x\,e^x-2\,e^x+c_1\,x+c_2$

Therefore $y=xe^x-3e^x+\dfrac{c_1\,x^2}{2}+c_2\,x+c_3$.

Ex. 11. *Solve :* $\dfrac{d^2y}{dx^2}=x^2\sin x$

Integrating, $\dfrac{dy}{dx}=-x^2\cos x+2\int x\cos x\,dx+c_1$

$$=-x^2\cos x+2\,x\sin x+2\cos x+c_1$$

$\therefore\quad y=-x^2\sin x-4\,x\cos x+6\sin x+c_1\,x+c_2$

EXERCISE VII (B)

Solve :

1. $\dfrac{d^2y}{dx^2}=x+\sin x.$
2. $\dfrac{d^2y}{dx^2}=x\,e^x.$
3. $\dfrac{d^2y}{dx^2}\cos^2 x=1$
4. $x^3\,\dfrac{d^3y}{dx^3}=1.$
5. $\dfrac{d^2y}{dx^2}=\dfrac{a}{x}$
6. $\dfrac{d^3y}{dx^3}\operatorname{cosec}^2 x=1.$
7. $\dfrac{d^2y}{dx^2}(x^2+a^2)^{1/2}=x.$
8. $x^2\dfrac{d^2y}{dx^2}=\log x.$

7·6. Equation of the form $\dfrac{d^2y}{dx^2}=f(y)$.

Multiplying both sides by $2\,\dfrac{dy}{dx}$, the equation becomes

$$2 \frac{dy}{dx} \frac{d^2y}{dx^2} = 2 f(y) \frac{dy}{dx};$$

and on integration with respect to x we get

$$\left(\frac{dy}{dx}\right)^2 = 2 \int f(y) \frac{dy}{dx} dx = 2 \int f(y)\, dy + c_1$$

or
$$\frac{dy}{[2 \int f(y)\, dy + c_1]^{1/2}} = dx$$

or
$$\int \frac{dy}{[2 \int f(y)\, dy + c_1]^{1/2}} = x + c_2.$$

Ex. 12. *Solve* : $\frac{d^2y}{dx^2} = \sec^2 y \tan y$, *given that* $y=0$ *and* $\frac{dy}{dx} = 1$ *when* $x=0$.

Multiplying by $2 \frac{dy}{dx}$ we get

$$2 \frac{dy}{dx} \frac{d^2y}{dx^2} = 2 \sec^2 y \tan y \frac{dy}{dx}.$$

Integrating $\left(\frac{dy}{dx}\right)^2 = \tan^2 y + 1 = \sec^2 y$, since when $x=0$, $y=0$ and $\frac{dy}{dx} = 1$, hence constant $=1$.

Therefore $\cos y\, dy = dx$.
Integrating again $\sin y = x + c_2$.
Now, when $x=0$, $y=0$, hence $c_2 = 0$.

$$\therefore \quad y = \sin^{-1} x.$$

EXERCISE VII (C)

Solve :

1. $\frac{d^2y}{dx^2} = y$.
2. $y^3 \frac{d^2y}{dx^2} = a$. (*Nagpur* 1958)
3. $\frac{d^2y}{dx^2} - a^2 y = 0$.
4. $\frac{d^2y}{dx^2} + \frac{a^2}{y} = 0$.

5. $\dfrac{d^2y}{dx^2}=y^3-y$; given that $\dfrac{dy}{dx}=0$ when $y=1$.

6. $\dfrac{d^2y}{dx^2}=e^{2y}$, given that $\dfrac{dy}{dx}=0, y=0$ when $x=0$.

7·7. An equation which does not contain y directly.

In such cases the order of equation may be depressed by assuming as a dependent variable the lowest differential coefficint which presents itself in the equation, Thus in the general equation

$$f\left(\frac{d^ny}{ax^n}, \ldots\ldots, \frac{dy}{dx}, x\right)=0,$$

when we write $\dfrac{dy}{dx}=p$ and $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}\ldots\ldots, \dfrac{d^ny}{dx^n}=\dfrac{d^{n-1}p}{dx^{n-1}}$

the order of the equation is lowered and it takes the form

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \ldots\ldots, p, x\right)=0.$$

which may possibly be solved for p. Suppose that we get

$$p=\frac{dy}{dx}=F(x),$$

then $\quad y=\int F(x)\, dx+c.$

Ex. 13. *Solve* : $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3=0.$ (*Agra* **1955**)

Substituting $\dfrac{dy}{dx}=p$, we get

$$\frac{dp}{dx}+p+p^3=0$$

or $\quad \dfrac{dp}{p(1+p^2)}=-dx$ or $\left(\dfrac{1}{p}-\dfrac{p}{1+p^2}\right)dp=-dx$

or $\quad \log p-\frac{1}{2}\log(1+p^2)=-x+c$

or $\quad \dfrac{p}{\sqrt{1+p^2}}=c_1 e^{-x}$

or $\quad p^2=\dfrac{c_1^2 e^{-2x}}{1-c_1^2 e^{-2x}}$

or $\quad p=\dfrac{dy}{dx}=\dfrac{c_1\, e^{-x}}{\sqrt{1-c_1^2\, e^{-2x}}}$

$\therefore \quad y=-\sin^{-1}(c_1\, e^{-x})+c_2.$

Ex. 14. *Solve :* $(1+x^2)\dfrac{d^2y}{dx^2}+1+\left(\dfrac{dy}{dx}\right)^2=0.$ (*Agra* 1957)

Writing $\dfrac{dy}{dx}=p$, and $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$, the equation is reduced to

$$\frac{dp}{1+p^2}+\frac{dx}{1+x^2}=0$$

$$\therefore \quad \tan^{-1} p+\tan^{-1} x=c_1,$$

or $\quad \tan^{-1}\dfrac{p+x}{1-px}=c_1 \quad$ or $\quad \dfrac{p+x}{1-px}=\tan c_1=A$

or $\quad p=\dfrac{A-x}{1-xA}=-\dfrac{1}{A}\left(\dfrac{Ax+1-A^2-1}{1+xA}\right)$

or $\quad \dfrac{dy}{dx}=-\dfrac{1}{A}\left(1-\dfrac{1+A^2}{1+xA}\right)$

or $\quad y=-\dfrac{1}{A}\,x+\dfrac{1+A^2}{A^2}\log(1+xA)+B$

$$=-\frac{1}{A}\,x+\left(1+\frac{1}{A^2}\right)\log(1+xA)+B$$

Writing $c=-1/A$, the integral is

$$y=cx+(1+c^2)\log(x-c)+c_1.$$

EXERCISE VII (D)

Solve :

1. $\dfrac{d^2y}{dx^2}=x\dfrac{dy}{dx}.$

2. $\dfrac{d^2y}{dx^2}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

3. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^x.$

4. $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=0.$

5. $x^2\dfrac{d^3y}{dx^3}-4x\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}=4.$

6. $\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x(a^2-x^2)}\dfrac{dy}{dx}=\dfrac{x^2}{a(a^2-x^2)}.$

7. $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+ax=0.$ 8. $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=ax.$

9. $x\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-\dfrac{dy}{dx}=0.$ 10. $x\dfrac{d^3y}{dx^3}-x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$

11. $\dfrac{dy}{dx}-x\dfrac{d^2y}{dx^2}=f\left(\dfrac{d^2y}{dx^2}\right)$ 12. $x\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)=x$

13. $(a^2-x^2)\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x}\dfrac{dy}{dx}+\dfrac{x^2}{a}=0$

7·8. The Equations which do not contain x directly. In such cases p is substitued for $\dfrac{dy}{dx}$. Then writing

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy},\ \frac{d^3y}{dx^3}=p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2, \text{ etc.}$$

the equation of the type

$$f\left(\frac{d^ny}{dx^n},\ldots\ldots\ldots\ldots\ldots,\frac{dy}{dx},y\right)=0$$

is reduced to $f\left(\dfrac{d^{n-1}p}{dy^{n-1}},\ldots\ldots\ldots\ldots\ldots,p,y\right)=0,$

which being an equation of the $(n-1)^{th}$ order between y and p may possibly be solved for p. Thus if we get

$$p=\frac{dy}{dx}=f(y),$$

then a solution of the original equation is

$$\int\frac{dy}{f(y)}=x+c.$$

Ex. 15. *Solve* : $y(1-\log y)\dfrac{d^2y}{dx^2}+(1+\log y)\left(\dfrac{dy}{dx}\right)^2=0.$

(*Agra* 1958)

Writing $\dfrac{dy}{dx}=p$ and $\dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$, the equation is reduced to

$$y(1-\log y)\,p\frac{dp}{dy}+(1+\log y)\,p^2=0$$

i.e. $$\frac{dp}{p}=\frac{(1+\log y)dy}{y\,(\log y-1)}$$

Now writing $\log y=t$ and therefore $\frac{dy}{y}=dt$ on the R. H. S., the equation becomes

$$\frac{dp}{p}=\frac{1+t}{t-1}\,dt=\left(1+\frac{2}{t-1}\right)dt$$

Integrating $\log p=t+2\log(t-1)+\text{const.}$

$$=\log y+\log(\log y-1)^2+\text{const.}$$

or $$p=\frac{dy}{dx}=c_1\,y(\log y-1)^2$$

or $$\frac{dy}{y\,(\log y-1)^2}=c_1\,dx$$

or $$-\frac{1}{\log y-1}=c_1\,x+c_2$$

or $$\log y-1=\frac{1}{-c_1\,x-c_2}=\frac{1}{a_1\,x+a_2};$$

Ex. 16. *Solve* : $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\log y.$

Substituting p for $\frac{dy}{dx}$ and $p\,\frac{dp}{dy}$ for $\frac{d^2y}{dx^2}$, the equation becomes $p\,\frac{dp}{dy}-\frac{p^2}{y}=y\log y.$

Now putting $p^2=v$ and therefore $p\,\frac{dp}{dy}=\frac{1}{2}\,\frac{dv}{dy}$, this becomes

$$\frac{dv}{dy}-\frac{2v}{y}=2y\log y$$

This is a linear equation and integrating we get

$$v=p^2=y^2\log^2 y+c_1\,y^2$$

or $$p=\frac{dy}{dx}=y\sqrt{\log^2 y+c_1}$$

whence $$\frac{dy}{y\sqrt{\log^2 y+c_1}}=dx.$$

Now if we substitute $\log y=t$ the equation becomes

$$\frac{dt}{\sqrt{t^2+c_1}}=dx$$

or $\log(t+\sqrt{t^2+c_1})=x+\log c_2$

or $t+\sqrt{t^2+c_1}=c_2\, e^x$

whence $t^2+c_1=t^2+c_2{}^2\, e^{2x}-2t\, c_2\, e^x$

or $t=\log y=A\, e^x+B\, e^{-x}$.

Ex. 17. *Solve* : $\left(\frac{dy}{dx}\right)^2-y\,\frac{d^2y}{dx^2}=n\left\{\left(\frac{dy}{dx}\right)^2+a^2\left(\frac{d^2y}{dx^2}\right)\right\}^{1/2}$

Putting $\frac{dy}{dx}=p$, and $\frac{d^2y}{dx^2}=p\,\frac{dp}{dy}$, the equation becomes

$$p^2-y\,p\,\frac{dp}{dy}=n\left\{p^2+a^2\,p^2\left(\frac{dp}{dy}\right)^2\right\}^{1/2}$$

or $p=y\frac{dp}{dy}+n\sqrt{1+a^2\left(\frac{dp}{dy}\right)^2}$

This is Clairaut's form ; hence the solution is

$$p=cy+n\sqrt{1+a^2\,c^2}$$

or $\frac{dy}{cy+n\sqrt{1+a^2\,c^2}}=dx$

or $\frac{1}{c}\log\{cy+n\sqrt{1+a^2\,c^2}\}=x+c_1$

or $cy+n\sqrt{1+a^2\,c^2}=c_2\,e^{cx}$.

EXERCISE VII (E)

Solve :

1. $\frac{d^2y}{dx^2}+y\,\frac{dy}{dx}=0.$

2. $y\,\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1.$

3. $y\,\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}=0.$

4. $\frac{d^2y}{dx^2}+2\,\frac{dy}{dx}+4\left(\frac{dy}{dx}\right)^2=0$

5. $\frac{d^2y}{dx^2}=a\left(\frac{dy}{dx}\right)^2$

6. $1+\left(\frac{dy}{dx}\right)^2+y\,\frac{d^2y}{dx^2}=0.$

7. $y\,\frac{d^2y}{dx^2}+\left\{\left(\frac{dy}{dx}\right)^2+a^2\left(\frac{d^2y}{dx^2}\right)^2\right\}^{\frac{1}{2}}=\left(\frac{dy}{dx}\right)^2.$

7·9. The equation of the form $f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, x\right)$ where differential coefficients differ in order by 1.

In such cases if we write $q=\frac{d^{n-1}y}{dx^{n-1}}$, the equation is reduced to

$$f\left(\frac{dq}{dx}, q, x\right)=0.$$

which may be integrated for q, being an equation of the first order between q and x.

Thus if $q=\frac{d^{n-1}y}{dx^{n-1}}=F(x)$

we can find the value of y by the method of successive integration

Ex. 18. *Solve* : $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0.$

Writing $\frac{dy}{dx}=q$, the given equation becomes

$$x\frac{dq}{dx}+q=0 \quad \text{or} \quad \frac{dq}{q}+\frac{dx}{x}=0,$$

giving $qx=c_1$ or $dy=\frac{c_1}{x}dx$

$\therefore \quad y=c_1 \log x+c_2.$

EXERCISE VII (F)

Solve :

1. $a\frac{d^2y}{dx^2}=\frac{dy}{dx}$.

2. $a^2\frac{d^2y}{dx^2}\frac{dy}{dx}=x.$

3. $\frac{d^3y}{dx^3}\frac{d^2y}{dx^2}=2.$

4. $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2+1.$

5. $a\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

6. $\frac{d^2y}{dx^2}=a^2+k^2\left(\frac{dy}{dx}\right)^2.$

7. $a^2\left(\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{dy}{dx}\right)^2.$

8. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0.$

9. $\frac{dy}{dx}=x\frac{d^2y}{dx^2}+\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$.

10. $\left(\frac{d^ny}{dx^n}\right)^2=4\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$.

7·10. The equation of the form $f\left(\frac{d^ny}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}, x\right)=0$ when differential coefficients differ in order by 2.

In such cases if we write $q=\frac{d^{n-2}y}{dx^{n-2}}$, the equation is reduced to

$$f\left(\frac{d^2q}{dx^2}, q, x\right)=0$$

from which q may be found. Thus if

$$q=\frac{d^{n-2}y}{dx^{n-2}}=F(x),$$

we can find the value of y by the method of successive integration.

Ex. 19. *Solve*: $\frac{d^4y}{dx^4}-a^2\frac{d^2y}{dx^2}=0.$

Writing $\frac{d^2y}{dx^2}=q$, it reduces to

$$\frac{d^2q}{dx^2}-a^2q=0$$

On integration, $q=c\sinh(ax+c_2)$

$$\frac{dy}{dx}=(c/a)\sinh(ax+c_2)+c_3$$

$$y=(c/a^2)\cosh(ax+c_2)+c_3x+c_4$$
$$y=c_1\cosh(ax+c_2)+c_3x+c_4.$$

Ex. 20. *Solve*: $\frac{d^ny}{dx^n}+\frac{d^{n-2}y}{dx^{n-2}}=8\cos 3x.$

Substituting $\frac{d^{n-2}y}{dx^{n-2}}=q$, the equation becomes

$$\frac{d^2q}{dx^2}+q=8\cos 3x$$

Integrating

$$q=\frac{d^{n-2}y}{dx^{n-2}}=c_1 \cos x+c_2 \sin x-\cos 3x.$$

Integrating $n-2$ times this gives

$$y=a_1 \cos x+a_2 \sin x-\frac{1}{3^{n-2}} \cos \{3x-\tfrac{1}{2}\pi (n-2)\}$$

$$+a_3 x^{n-3}+\dots\dots+a_{n-1} x+a_n$$

EXERCISE VII (G)

Solve :

1. $a^2 \dfrac{d^4y}{dx^4}=\dfrac{d^2y}{dx^2}$

2. $\dfrac{d^4y}{dx^4}+a^2\dfrac{d^2y}{dx^2}=0.$

3. $\dfrac{d^5y}{dx^5}-n^2\dfrac{d^3y}{dx^3}=e^{ax}.$

4. $x^2 \dfrac{d^4y}{dx^4}+a^2\dfrac{d^2y}{dx^2}=0.$

5. $x^2 \dfrac{d^4y}{dx^4}=\lambda \dfrac{d^2y}{dx^2}$

7·11. Homogeneous Equations.

In chapter II we have considered equations of first order and first degree which are homogeneous. In chapter VI homogeneous linear equations were dealt with. Now we shall consider the general case of homogeneous equations.

If y is of n dimensions and x of one dimension, then $\dfrac{dy}{dx}$ is of $n-1$ dimensions, $\dfrac{d^2y}{dx^2}$ is of $n-2$ dimensions and so on.

When all the terms of a differential equation are of the same dimensions, it is called a **homogeneous equation.**

In such cases suitable transformations are made to lower the order of the equations.

Case I. *If x, y both be of one dimension.* In such cases we substitute $y=xz$ and $x=e^{\theta}$. Therefore

$$\frac{dy}{dx}=z+x\frac{dz}{dx}=z+x.\frac{dz}{d\theta}\cdot\frac{d\theta}{dx}=z+\frac{dz}{d\theta},$$

and $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(z+\dfrac{dz}{d\theta}\right)=\left(\dfrac{d^2z}{d\theta^2}+\dfrac{dz}{d\theta}\right)e^{-\theta}$, and so on.

On substitution the equation will be reduced into one between z and θ. The equation being homogeneous will contain the same index of θ in the exponential for each term of the equation and this factor, therefore may be cancelled. The resulting equation will be of a class already discussed.

Ex. 21. *Solve :* $x^4 \dfrac{d^2y}{dx^2}=\left(y-x\dfrac{dy}{dx} \right)^3.$

Making the substitutions given above, we have

$$\frac{d^2z}{d\theta^2}+\frac{dz}{d\theta}=-\left(\frac{dz}{d\theta} \right)^3$$

If we put $\dfrac{dz}{d\theta}=v$, then the equation becomes

$$\frac{dv}{d\theta}=-v\ (1+v^2)$$

On integration we get

$$\frac{v}{\sqrt{(1+v^2)}}=c_1\ e^{-\theta}$$

or $$dz=\frac{c_1\,e^{-\theta}}{\sqrt{\left(1-c_1^2\,e^{-2\theta} \right)}}\ d\theta.$$

Integrating $z=-\sin^{-1}\left(c_1\,e^{-\theta} \right)+c_2$

or $y=-x\sin^{-1}(c_1/x)+c_2\,x.$

Case II. *If y is of n dimensions and x of one.* In such cases we substitute

$$x=e^{\theta}\ ,\ y=x^n z=z\,e^{n\theta}$$

$$\frac{dy}{dx}=nx^{n-1}z+x^n\frac{dz}{dx}=nx^{n-1}z+x^n\frac{dz}{d\theta}\cdot\frac{d\theta}{dx}$$

$$=\left(nz+\frac{dz}{d\theta} \right)e^{(n-1)\theta},$$

and $$\frac{d^2y}{dx^2}=\left\{ \frac{d^2y}{d\theta^2}+(2n-1)\ \frac{dz}{d\theta}+n\ (n-1)\ z \right\}e^{(n-2)\theta}\ .$$

and so on.

In this case also every term of the resulting differential equation will contain the same index of θ in the exponential and

therefore can be removed. In the new form of the equation the independent variable is explicitly absent and is therefore of a class already discussed.

Ex. 22. *Solve* : $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x^2\left(\frac{dy}{dx}\right)_4-y^2$

It is a homogeneous equation if y is considered of -1 dimension and x of one. Clearly $\frac{dy}{dx}$ is of -2 dimensions and $\frac{d^2y}{dx^2}$ is of -3 dimensions. Therefore first term is of -2 dimensions and similarly others. Here we substitute $x=e^{\theta}$, $y=x^{-1}z=ze^{-\theta}$ and therefore

$$\frac{dy}{dx}=\left(\frac{dz}{d\theta}-z\right)e^{-2\theta} \;:$$

and $\frac{d^2y}{dx^2}=\left(\frac{d^2z}{d\theta^2}-3\frac{dz}{d\theta}+2z\right)e^{-3\theta}$

The equation is reduced to

$$\frac{d^2z}{d\theta^2}+(2z-1)\frac{dz}{d\theta}-\left(\frac{dz}{d\theta}\right)^2=0.$$

Now put $\frac{dz}{d\theta}=q$ and $\frac{d^2z}{{}_2}=q\frac{dq}{dz}$

the equation becomes

$$\frac{dq}{dz}+(2z-1)-q=0$$

or $\qquad \frac{dq}{dz}-q=1-2z.$

It is a linear equatior. .nd its integral is

$$q\,e^{-}=\int\ {}^{-z}(1-2z)\,dz+c_1$$

or $\qquad q=1+2z+c_1e^z$

or $\qquad d\theta=\frac{dz}{1+2z+c_1e^z}$

whence $\theta=\int\frac{dz}{1+2z+c_1e^z}+c_2.$

EXERCISE VII (H)

Solve :

1. $n\,x^3 \dfrac{d^2y}{dx^2} = \left(y - x\dfrac{dy}{dx}\right)$

2. $xy\,\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 = 3y\dfrac{dy}{dx}$

3. $2x^2\,y\,\dfrac{d^2y}{dx^2} + y^2 = x^2\left(\dfrac{dy}{dx}\right)^2$

4. $x^2\,\dfrac{d^2y}{dx^2} = \left\{m\,x^2\left(\dfrac{dy}{dx}\right)^2 + n\,y^2\right\}^{\frac{1}{2}}$

5. $x^4\,\dfrac{d^2y}{dx^2} = (x^3 + 2\,xy)\,\dfrac{dy}{dx} - 4\,y^2$.

6. $x^4\,\dfrac{d^2y}{dx^2} - x^3\dfrac{dy}{dx} = x^2\left(\dfrac{dy}{dx}\right)^2 - 4y^2$

7. $x^2\,\dfrac{d^2y}{dx^2} + 4y^2 - 6y = x^4\left(\dfrac{dy}{dx}\right)^2$.

Miscellaneous Exercise on Chapter VII

Solve :

1. $y_2 = e^y$
2. $y_2 + a^2y = 0$
3. $ay_3 = y_2$
4. $x^2\,y_4 + 1 = 0$
5. $y_3 = \sin^2 x$
6. $y_2 = (ay)^{-1/2}$
7. $(1+x^2)\,y_2 + 3xy_1 + y = 0$ (*Agra* 1953)
8. $-ay_2 = (1+y_1^2)^{3/2}$
9. $\sin^3 y\,\dfrac{d^2y}{dx^2} = \cos y$.
10. $e^{\frac{1}{2}x^2}(xy_2 - y_1) = x^3$
11. $(1-x^2)\,y_2 - xy_1 = 2$ (*Agra* 1952)
12. $2xy_3\,y_2 = y_2^2 - a^2$
13. $yy_2 + (y_1^2 + a^2\,y_2^2)^{1/2} = y_1^2$.
14. Show that the equation

$$(x^3 - 4x)\,\frac{d^3y}{dx^3} + (9x^2 - 4)\,\frac{d^2y}{dx^2} + 18\,x\,\frac{dy}{dx} + 6y = 6$$

is exact and solve it. (*I. A. S.* 1952)

CHAPTER VIII

LINEAR EQUATIONS OF SECOND ORDER

8 1. In this chapter we shall consider linear equations of second order of the form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R, \qquad (1)$$

where P, Q and R are functions of x.

No general method of solving such equations can be given. However we shall consider some particular cases in which the integral can be found. Suffixes, as usual, will be used to denote d ifferentiations.

8·2. Complete solution in terms of a known integral belonging to the complementary function. If an integral included in the complementary function of a second order linear differential equation be known its order can be depressed and the complete solution can be found.

Let $y=u$ be an integral in the complementary function of (1). Put $y=uv$ so that $y_1=u_1 v+v_1 u$ and $y_2=u_2 v+2u_1 v_1+uv_2$.

Substituting these values of y, y_1 and y_2 in (1), we get

$$uv_2+(2u_1+Pu)\,v_1+(u_2+Pu_1+Qu)\,v=R \qquad (2)$$

The coefficient of v vanishes since $y=u$ is a solution of

$$y_2+Py_1+Qy=0.$$

Thus (2) becomes

$$v_2+\left(2\frac{u_1}{u}+P\right)v_1=\frac{R}{u}.$$

This is a linear equation in v_1. Hence v_1 can be determined ; and then

$$y=u\int v_1\,dx+c_1 u$$

is the integral of equation (1), the other constant of integration will occur in the expression for v_1. Thus we get the complete primitive.

Note. *Sometimes t e complementary function can be found by inspection, if the following rules are o rved.*

$y=x$ is a part of C. F. if $P+Qx=0$,

$y=e^x$ is a part of C. F. $1+P+Q=0$,

$y=e^{ax}$ is a part of C. F. if $1+P/a+Q/a^2=0$,

$y=e^{-x}$ is a part of C. F. if $1-P+Q=0$,

$y=x^2$ is a part of C. F. if $2+2Px+Qx^2=0$.

Ex. 1. *Solve :* $x\dfrac{d^2y}{dx^2}-(2x-1)\dfrac{dy}{dx}+(x-1)y=0.$

(*Agra* 1955, *Alld.* '58)

Here, the sum of the coefficients being zero, e^x is obviously a solution of the given equation.

Substituting $y=v\,e^x$, $\dfrac{dy}{dx}=v\,e^x+\dfrac{dv}{dx}e^x$

and $\dfrac{d^2y}{dx^2}=v\,e^x+2\dfrac{dv}{dx}e^x+\dfrac{d^2v}{dx^2}e^x$

in the equation, it becomes

$$x\frac{d^2v}{dx^2}+\frac{dv}{dx}=0.$$

This is exact, hence integrating

$$x\frac{dv}{dx}=c_1 \quad \text{or} \quad \frac{dv}{dx}=\frac{c_1}{x}.$$

Therefore $v=c_1\log x+c_2$.

Hence complete solution is

$$y=v\,e^x=e^x(c_1\log x+c_2).$$

Ex. 2. *Solve :* $\sin^2 x\,\dfrac{d^2y}{dx^2}=2y.$

(*Agra* 1956, 1962, *Jaipur* '49, '52, *Andhra* '53)

Here $y=\cot x$ is a solution of $\dfrac{d^2y}{dx^2}-2y\,\text{cosec}^2 x=0.$

Substituting $y=v\cot x$ and therefore

$$\frac{dy}{dx}=\cot x\frac{dv}{dx}-v\,\text{cosec}^2 x$$

and $\dfrac{d^2y}{dx^2}=\cot x\,\dfrac{d^2v}{dx^2}-2\,\text{cosec}^2 x\,\dfrac{dv}{dx}+2\,\text{cosec}^2 x\cot x,$

the original equation pecomes

$$\frac{d^2v}{dx^2}-\frac{2}{\sin x\cos x}\frac{dv}{dx}=0.$$

Substituting p for $\dfrac{dv}{dx}$ this becomes

$$\frac{dp}{dx}-\frac{2}{\sin x \cos x}\,p=0$$

$$\text{or}\quad \frac{dp}{p}=\frac{2dx}{\sin x \cos x}$$

Integrating $\log p=2 \log \tan x+\log c_1$

whence $p=\dfrac{dv}{dx}=c_1 \tan^2 x=c_1 (\sec^2 x-1)$

Therefore $v=c_1 \tan x-c_1 x+c_2$.

Hence the complete solution is

$$y=v \cot x=c_1-c_1 x \cot x+c_2 \cot x.$$

Ex. 3. *Solve :* $(1-x^2) \dfrac{d^2y}{dx^2}+x \dfrac{dy}{dx}-y=x (1-x^2)^{3/2}$

(*I. A. S.* 1951, *Jaipur* 1951, *Punjab* 1956)

Here $y=x$ is obviously a solution of the equation

$$(1-x^2) \frac{d^2y}{dx^2}+x \frac{dy}{dx}-y=0.$$

Substituting $y=vx$ and therefore $\dfrac{dy}{dx}=v+x \dfrac{dv}{dx}$

and $$\frac{d^2y}{dx^2}=x \frac{d^2v}{dx^2}+2 \frac{dv}{dx},$$

the original equation becomes

$$\frac{d^2v}{dx^2}+\left(\frac{2}{x}+\frac{x}{1-x^2} \right) \frac{dv}{dx}=(1-x^2)^{1/2}.$$

Substituting for $\dfrac{dv}{dx}$ this becomes

$$\frac{dp}{dx}+\left(\frac{2}{x}+\frac{x}{1-x^2} \right) p=(1-x^2)^{1/2}$$

This is linear and the integrating factor is $x^2/(1-x^2)^{1/2}$,

Therefore $p \dfrac{x^2}{\sqrt{(1-x^2)}}=\dfrac{1}{3} x^3+c_1$

$$\text{or}\quad p=\frac{dv}{dx}=\frac{1}{3} x\sqrt{1-x^2}+\frac{c_1}{x^2} \sqrt{1-x^2}.$$

Integrating again we easily get

$$v=-\frac{1}{9}(1-x^2)^{3/2}+c_1\left(\sin^{-1}x+\frac{\sqrt{1-x^2}}{x}\right)+c_2.$$

Hence the complete solution is

$$y=vx=-\tfrac{1}{9}x(1-x^2)^{3/2}+c_1x(\sin^{-1}x+\sqrt{1-x^2})+c_2x.$$

Ex. 4. **Solve* :

$$(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x. \quad (\textit{Agra } 1947, 1952)$$

Here $y=e^{2x}$ is a solution of the equation

$$(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=0.$$

Substituting $y=v\,e^{2x}$ and therefore

$$\frac{dy}{dx}=\left(\frac{dv}{dx}+2v\right)e^{2x} \text{ and } \frac{d^2y}{dx^2}=\left(\frac{d^2v}{dx^2}+4\frac{dv}{dx}+4v\right)e^{2x},$$

the original equation becomes

$$(x+2)\frac{d^2v}{dx^2}+(2x+3)\frac{dv}{dx}=(x+1)\,e^{-x}.$$

Substituting p for $\frac{dv}{dx}$, this becomes

$$\frac{dp}{dx}+\frac{(2x+3)}{x+2}\,p=\frac{(x+1)\,e^{-x}}{x+2}.$$

This is a linear equation and the integrating factor is
$e^{2x}(x+2)^{-1}$

$$\text{Therefore}\quad p\,e^{2x}(x+2)^{-1}=\int\frac{x+1}{(x+2)^2}\,e^x\,dx+c_1$$

$$=\frac{e^x}{x+2}+c_1.$$

$$\text{or}\quad p=\frac{dv}{dx}=e^{-x}+c_1\,e^{-2x}(x+2)$$

Integrating again $\quad v=-e^{-x}-\tfrac{1}{4}c_1(2x+5)\,e^{-2x}+c_2.$
Hence the complete solution is

$$y=v\,e^{2x}=-e^x-\tfrac{1}{4}c_1(2x+5)+c_2\,e^{2x}.$$

Ex. 5 *Solve* : $\frac{d^2y}{dx^2}-\cot x\,\frac{dy}{dx}-(1-\cot x)\,y=e^x\sin x$

(*Jaipur*, 1957)

*Can also be done easily by the method of Art. 8·5.

Hence the sum of the coefficients being zero, e^x is obviously a solution of the given equation.

Substituting $y=v\,e^x$, we get

$$\frac{d^2v}{dx^2}+(2-\cot x)\frac{dv}{dx}=\sin x.$$

Let $\dfrac{dv}{dx}=p$ then the equation is

$$\frac{dp}{dx}+(2-\cot x)\,p=\sin x$$

Integrating factor of this linear equation is

$$e^{\int(2-\cot x)\,dx}=e^{2x-\log\sin x}=\frac{e^{2x}}{\sin x}$$

$$\therefore\quad p\cdot\frac{e^{2x}}{\sin x}=\int e^{2x}\,dx+c_1$$

or $\quad p=\frac{1}{2}\sin x+c_1\,e^{-2x}\sin x$

$$\therefore\quad v=-\tfrac{1}{2}\cos x+c_1\int e^{-2x}\sin x\,dx+c_2$$

$$=-\tfrac{1}{2}\cos x-\tfrac{1}{5}c_1\,e^{-2x}(\cos x+2\sin x)+c_2$$

$\therefore$ The complete primitive is

$$y=ve^x=-\tfrac{1}{2}e^x\cos x-\tfrac{1}{5}c_1\,e^{-x}(\cos x+2\sin x)+c_2\,e^x.$$

Ex. 6. *Solve* : $(x\sin x+\cos x)\dfrac{d^2y}{dx^2}-x\cos x\dfrac{dy}{dx}+y\cos x=0,$ *of which* $y=x$ *is a solution.* (*Agra* 1942, 1949)

Substituting x for y in the given equation we get

$$\frac{d^2v}{dx^2}+\frac{2x\sin x+2\cos x-x^2\cos x}{x^2\sin x+x\cos x}\frac{dv}{dx}=0.$$

Substituting p for $\dfrac{dv}{dx}$ this becomes

$$\frac{dp}{dx}+\frac{2(x\sin x+\cos x)-x^2\cos x}{x(x\sin x+\cos x)}\,p=0$$

or $$\frac{dp}{dx}+\left(\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right)p=0$$

or $$\frac{dp}{p}+\left(\frac{2}{x}-\frac{x\cos x}{x\sin x+\cos x}\right)dx=0.$$

Integrating, $\dfrac{p\,x^2}{x\sin x+\cos x}=c$

or $p=\dfrac{dv}{dx}=c\left(\dfrac{\sin x}{x}+\dfrac{\cos x}{x^2}\right)$

Integrating again

$$v=c\left(\int\frac{\sin x}{x}\,dx-\frac{\cos x}{x}-\int\frac{\sin x}{x}\,dx\right)+c_2=c_1\,x^{-1}\cos x+c_2$$

Hence the complete integral is

$$y=vx=c_1\cos x+c_2\,x.$$

Ex. 7. *Solve :* $\dfrac{d^2y}{dx^2}+\left(1+\dfrac{2}{x}\cot x-\dfrac{2}{x^2}\right)y=x\cos x,$

given that $\dfrac{\sin x}{x}$ *is a C F.* (*I. A. S.* 1952)

Substituting $y=\dfrac{v\sin x}{x},\ \dfrac{dy}{dx}=\dfrac{v_1\sin x}{x}+v\left(\dfrac{x\cos x-\sin x}{x^2}\right)$

and $\dfrac{d^2y}{dx^2}=\dfrac{d^2v}{dx^2}\,\dfrac{\sin x}{x}+2\,\dfrac{dv}{dx}\left(\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}\right)$

$$+v\left(-\frac{\sin x}{x}-\frac{2\cos x}{x^2}+\frac{2\sin x}{x^3}\right)$$

the given equation reduces to

$$\frac{d^2v}{dx^2}+2\left(\cot x-\frac{1}{x}\right)\frac{dv}{dx}=x^2\cot x.$$

Putting $\dfrac{dv}{dx}=q$, it becomes

$$\frac{dq}{dx}+2\left(\cot x-\frac{1}{x}\right)q=x^2\cot x.$$

The equation is linear and the integrating factor is $\dfrac{\sin^2 x}{x^2}$.

$$\therefore\ q\,\frac{\sin^2 x}{x^2}=\int\tfrac{1}{2}\sin 2x\,dx+c_1$$

$$=-\tfrac{1}{4}\cos 2x+c_1=-\tfrac{1}{4}+\tfrac{1}{2}\sin^2 x+c_1$$

or $\dfrac{dv}{dx}=-\tfrac{1}{4}\,x^2\operatorname{cosec}^2 x+\tfrac{1}{2}\,x^2++c_1\,x^2\operatorname{cosec}^2 x$

$v=\frac{1}{6}x^3+(c_1-\frac{1}{4})\left[-x^2\cot x+2x\log\sin x-2\int\log\sin x\,dx\right]+c_2$

$\therefore$ Complete primitive is $y=\dfrac{v\sin x}{x}$, where v is the expression given above.

EXERCISE VIII (A)

Solve :

1. $\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=x.$ (*Agra* 1949)

2. $x\dfrac{d^2y}{dx^2}-(3+x)\dfrac{dy}{dx}+3y=0.$

3. $x\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}=y+e^x.$

4. $(x+1)\dfrac{d^2y}{dx^2}-2(x+3)\dfrac{dy}{dx}+(x+5)y=e^x.$ (*Nagpur* 1953)

5. $(3-x)\dfrac{d^2y}{dx^2}-(9-4x)\dfrac{dy}{dx}+(6-3x)y=0.$ (*Allahabad* 1953)

6. $\boxed{\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=X.}$

7. $\dfrac{d^3y}{dx^3}-x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+xy=0.$

8. $x^2\dfrac{d^3y}{dx^3}+x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

9. $x^2\dfrac{d^2y}{dx^2}-(x^2+2x)\dfrac{dy}{dx}+(x+2)y=x^3e^x$, of which $y=x$ is a solution. (*Agra* 1950, 1954)

10. $\dfrac{d^2y}{dx^2}-ax\dfrac{dy}{dx}+a^2(x-1)=0$, of which $y=e^{ax}$ is a solution.

11. $(2x^3-a)\dfrac{d^2y}{dx^2}-6x^2\dfrac{dy}{dx}+6xy=0$, of which $y=x$ is a solution.

8·3. Removal of the first derivative. Normal form.

If an integral included in the complementary function is not obvious by inspection or can not be determined easily, it is sometimes useful to reduce (1) Art. 8·1 to the normal form in which the term containing the first derivative is missing. To do so we put $y=uv$, proceeding as in Art. 8·1, we get

$$v_2+\left(\frac{2u_1}{u}+P\right)v_1+\left(\frac{u_2}{u}+\frac{Pu_1}{u}+Q\right)v=\frac{R}{u}. \tag{3}$$

Let us choose u from the relation $\frac{2u_1}{u}+P=0$.

$$\therefore \quad u=e^{-\frac{1}{2}\int P\,dx} .$$

The equation (3) becomes.

$$\frac{d^2y}{dx^2}+I\,v=S, \tag{4}$$

where

$$^{*}I=Q-\frac{1}{2}\,\frac{dP}{dx}-\frac{1}{4}\,P^2 \text{ and } S=R\,e^{\frac{1}{2}\int P\,dx}$$

If the value of I is a constant or a constant divided by x^2 equation (4) becomes readily integrable.

$\frac{d^2v}{dx^2}+I\,v=S$ is said to be the normal form of equation (1).

Ex. 8. *Solve* :

$$x^2\,\frac{d^2y}{dx^2}-2\,(x^2+x)\,\frac{dy}{dx}+(x^2+2x+2)\,y=0. \qquad (\textit{Agra } 1957)$$

The equation may be written in the form

$$\frac{d^2y}{dx^2}-2\left(1+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)y=0.$$

Here $P=-2\left(1+\frac{1}{x}\right)$, $Q=1+\frac{2}{x}+\frac{2}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P dx}=e^{\int (1+1/x)\,dx}=e^{x+\log x}=xe^x .$$

$$\therefore \quad I=Q-\tfrac{1}{2}\,\frac{dP}{dx}-\tfrac{1}{4}\,P^2=0.$$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}=0 \quad \text{or} \quad \frac{dv}{dx}=c_1 \quad \text{or} \quad v=c_1\,x+c_2$$

Therefore the general solution of the given equation is

$$y=u\,v=x\,e^x\,(c_1\,x+c_2)=e^x\,(c_1\,x^2+c_2\,x).$$

Ex. 9. *Solve* : $x\,\frac{d}{dx}\left(x\frac{dy}{dx}-y\right)-2x\frac{dy}{dx}+2y+x^2\,y=0.$

(*Rajasthan* 1957)

*Students should memorise the values of I and S.

On simplification the equation can be put in the form

$$\frac{d^2y}{dx^2}-\frac{2}{x}\frac{dy}{dx}+\left(1+\frac{2}{x^2}\right)y=0$$

Here $P=-\frac{2}{x}$, $Q=1+\frac{2}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P dx}=x$$

and $I=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2=1$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}+v=0 \quad \text{whence } v=c_1 \cos x+c_2 \sin x$$

$\therefore$ $y=x\,(c_1 \cos x+c_2 \sin x)$ is the complete primitive.

Ex. 10. *Solve* :

$$\frac{d^2y}{dx^2}+\frac{1}{x^{1/3}}\frac{dy}{dx}+\left(\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}\right)y=0.$$

(*Allahabad* 1942, *Jaipur* 1951)

Here $P=x^{-\frac{1}{3}}$, $Q=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{3}{4}x^{2/3}}$$

and $I=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}+\frac{1}{6}\frac{1}{x^{4/3}}-\frac{1}{4}\frac{1}{x^{2/3}}=-\frac{6}{x^2}.$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}-\frac{6v}{x^2}=0 \quad \text{or} \quad x^2\frac{d^2v}{dx^2}-6v=0.$$

This is a homogeneous linear equation and its solution is

$$v=c_1 x^3+c_2 x^{-2}.$$

Hence the complete primitive is

$$y=uv=e^{-\frac{3}{4}x^{2/3}}(c_1 x^3+c_2' x^{-2}).$$

Ex. 11. *Solve* : $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right)+y\cos^2 x=0$

(*I.A.S.* 1953)

On simplification the above equation becomes

$$\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+y=0$$

Here $P=-2\tan x$, $Q=1$

$$\therefore \quad u=e^{-\frac{1}{2}\int P\,dx}=\sec x.$$

$$I=Q-\tfrac{1}{2}\frac{dP}{dx}-\tfrac{1}{4}P^2=2,$$

$\therefore$ Reduced equation is

$$\frac{d^2v}{dx^2}+2v=0$$

Solution is $v=c_1\cos x\sqrt{2}+c_2\sin x\sqrt{2}$.

$\therefore \quad y=uv=\sec x\,(c_1\cos x\sqrt{2}+c_2\sin x\sqrt{2})$

Ex. 12. *Solve the equation :*

$$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)\,y=-3\,e^{x^2}\sin 2x.$$

(*Agra* 1933, 1941, 1950, 1955 ; *Luck.* 1951, 1956 ; *I.A.S.* 1956)

Here $P=-4x$, $Q=4x^2-1$;

$$\text{hence} \quad u=e^{-\frac{1}{2}\int P dx}=e^{\int 2x dx}=e^{x^2}$$

$$I=Q-\tfrac{1}{2}\frac{dP}{dx}-\tfrac{1}{4}P^2=4x^2-1-\tfrac{1}{2}(-4)-\tfrac{1}{4}\cdot 16x^2=1.$$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}+v=-3\,e^{x^2}\sin 2x\;e^{\frac{1}{2}\int -4x dx}=-3\sin 2x.$$

The solution of this equation is

$$v=c_1\cos x+c_2\sin x+\sin 2x.$$

Hence the complete primitive is

$$y=vu=e^{x^2}(c_1\cos x+c_2\sin x+\sin 2x).$$

EXERCISE VIII (B)

Solve :

1. $\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+4x^2\,y=0$ (*Agra* 1953)

2. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+n^2\,y=0.$ (*Agra* 1935)

3. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}=n^2\,y.$ (*Agra* 1949, *Luck.* 1951)

4. $\dfrac{d^2y}{dx^2}-2bx\dfrac{dy}{dx}+b^2x^2y=0$ (*Agra* 1956)

5. $\dfrac{d^2y}{dx^2}-2bx\dfrac{dy}{dx}+b^2x^2y=x$ (*Luck.* 1949)

6. $4x^2\dfrac{d^2y}{dx^2}+4x^5\dfrac{dy}{dx}+(x^8+6x^4+4)y=0$

7. $x^2\dfrac{d^2y}{dx^2}+(x-4x^2)\dfrac{dy}{dx}+(1-2x+4x^2)y=0$

8. $\dfrac{d^2y}{dx^2}-2\tan x\dfrac{dy}{dx}+5y=\sec x\,.\,e^x$ (*Agra* 1961)

9. $\dfrac{d^2y}{dx^2}-2\tan x\dfrac{dy}{dx}-(a^2+1)y=0.$

10. $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(n^2+\dfrac{2}{x^2}\right)y=0$

11. $\dfrac{d^2y}{dx^2}+2n\cot nx\dfrac{dy}{dx}+(m^2-n^2)y=0.$

12. $\dfrac{d^2y}{dx^2}-\dfrac{1}{\sqrt{x}}\dfrac{dy}{dx}+\dfrac{y}{4x^2}(-8+\sqrt{x}+x)=0$

13. $x^2\dfrac{d^2y}{dx^2}-2nx\dfrac{dy}{dx}+(n^2+n+a^2x^2)y=0$

14. $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(4x^2-3)y=e^{x^2}.$ (*Nagpur* 1961)

8·4. Change of the Independent Variable. Another method which sometimes is very useful in transforming the equation in an integrable form is that of *changing the independent variable.*

Let the linear equation of second order be

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Q=R,$$

where P, Q and R are functions of x and let the independent variable be changed from x to z, z, being a given function of x.

Since $\dfrac{dy}{dx}=\dfrac{dy}{dz}\dfrac{dz}{dx},$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right)=\frac{d}{dz}\left(\frac{dy}{dz}\frac{dz}{dx}\right)\frac{dz}{dx}=\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\frac{d^2z}{dx^2}$,

the original equation becomes

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2}+\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\frac{dy}{dz}+Qy=R$$

or $$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1 y=R_1 \qquad \ldots\ldots(2)$$

where $$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\ Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}.$$

P_1, Q_1, R_1 are functions of x as shown above but can be readily expressed as functions of z by the given relation between z and x.

If by equating $\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ to a constant quantity we find that P_1 also becomes constant then equation (2) is at once integrable. Since z is quite arbitrary, it may therefore be chosen to satisfy any assignable condition. Thus we may choose z to make the coefficient of $\frac{dy}{dx}$ vanish ; hence if we put $P_1=0$ or $\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0$

$$i.e.\quad z=\int e^{-\int P\,dx}\,dx.$$

and the value of Q_1 comes out to be a constant or a constant divided by z^2 then equation (2) becomes integrable. If Q_1 is a^2, then

$$a\frac{dz}{dx}=\sqrt{Q} \quad \therefore \quad az=\int\sqrt{Q}\,dx$$

Ex. 13. *Solve* : $\frac{d^2y}{dx^2}-(8e^{2x}+2)\frac{dy}{dx}+4e^{4x}y=e^{6x}$.

Putting $\frac{dz}{dx}=2e^{2x}$ *i.e.* $z=e^{2x}$ and $\frac{d^2z}{dx^2}=4e^{2x}$, the equation becomes

$$\frac{d^2y}{dz^2}-4\frac{dy}{dz}+y=\tfrac{1}{4}z.$$

Complementary function is $c_1\, e^{(2+\sqrt{3})z}+c_2\, e^{(2-\sqrt{3})\,z}$,

and particular integral$=\frac{1}{4}\,\dfrac{1}{1-4D+D^2}\,z=\frac{1}{4}\,(z+4)$.

Hence the solution is

$$y=c_1\, e^{(2+\sqrt{3})\,e^{2x}}+c_2\, e^{(2-\sqrt{3})\,e^{2x}}+\tfrac{1}{4}\,e^{2x}+1$$

Ex. 14. *Solve :* $\dfrac{d^2y}{dx^2}+\cot x\,\dfrac{dy}{dx}+4y\,\text{cosec}^2\,x=0.$

(*Raj.* 1956, *I.A.S.* 1951, 1959 ; *Punjab* 1957, *Agra* 1951, '55)

Let us choose z such that $\left(\dfrac{dz}{dx}\right)^2=4\,\text{cosec}^2\,x$, so that

$$dz=2\,\text{cosec}\,x\,dx \quad \text{or} \quad z=2\log\tan\tfrac{1}{2}x.$$

Hence by changing the independent variable from x to z, the equation becomes

$$\frac{d^2y}{dz^2}+y=0.$$

Integrating it we get

$$y=A\cos(z+B).$$

Hence the solution of the given equation is

$$y=A\cos(2\log\tan\tfrac{1}{2}x+B).$$

Ex. 15. *Solve :* $x^6\,\dfrac{d^2y}{dx^2}+3\,x^5\,\dfrac{dy}{dx}+a^2\,y=\dfrac{1}{x^2}.$

(*Allahabad* 1949, *Agra* 1953)

The equation may be written in the form

$$\frac{d^2y}{dx^2}+\frac{3}{x}\,\frac{dy}{dx}+\frac{a^2}{x^6}=\frac{1}{x^8}.$$

Let us choose z such that $\left(\dfrac{dz}{dx}\right)^2=\dfrac{a^2}{x^6}$

$$\text{or}\ \frac{dz}{dx}=\frac{a}{x^3} \quad \text{or} \quad z=-\frac{a}{2x^2}.$$

Here $\quad R_1=\dfrac{R}{(dz/dx)^2}=\dfrac{1}{x^8}\cdot\dfrac{x^6}{a^2}=\dfrac{1}{x^2\,a^2}=-\dfrac{2z}{a^3}.$

Change of the variable from x to z will now give

$$\frac{d^2y}{dx^2}+y=-\frac{2z}{a^3}$$

Here $C.F.$ is $c_1 \cos z+c_2 \sin z=c_1 \cos a/(2x^2)+c^2 \sin a/(2x^2)$,

and $$P.I.=\frac{1}{D^2+1}\left(-\frac{2}{a^3}z\right)=-\frac{2}{a^3}(1-D^2)z$$

$$=-\frac{2}{a^3}z=-\frac{2}{a^3}\left(-\frac{a}{2x^2}\right)=\frac{1}{a^2x^2}.$$

$\therefore$ $y=c_1 \cos [a/(2x^2)]+c_2 \sin [a/(2x^2)]+1/(a^2x^2)$.

Ex. 16. *Solve* : $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-4x^3y=8x^3 \sin x^2$.

(*Agra* 1934, *Jaipur* 1952)

The equation may be written in the form

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=8x^2 \sin x^2.$$

Find z such that $\left(\dfrac{dz}{dx}\right)^2=4x^2$

or $dz=2x\,dx$ or $z=x^2$,

Here $R_1=\dfrac{R}{(dz/dx)^2}=\dfrac{8x^2 \sin x^2}{4x^2}=2 \sin x^2=2 \sin z.$

Change of the variable from x to z will now give

$$\frac{d^2y}{dz^2}-y=2 \sin z.$$

Here $C.F.=c_1 e^z+c_2 e^{-z}=c_1 e^{x^2}+c_2 e^{-x^2}$

and $P.I.=\dfrac{1}{D^2-1}(2 \sin z)=-\sin z=-\sin x^2$

$\therefore$ $y=c_1 e^{x^2}+c_2 e^{-x^2}-\sin x^2$.

Ex. 17. *Solve* : $\dfrac{d^2y}{dx^2}\cos x+\dfrac{dy}{dx}\sin x-2y \cos^3 x=2\cos^5 x$.

(*Agra* 1960)

Dividing by $\cos x$, the equation becomes

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}-2y\cos^2 x=2\cos^4 x.$$

Find z such that $\left(\dfrac{dz}{dx}\right)^2=\cos^2 x$ or $dz=\cos x\,dx$ or $z=\sin x$.

Here $R_1=R\left(\dfrac{dz}{dx}\right)^{-2}=\dfrac{2\cos^4 x}{\cos^2 x}=2\cos^2 x=2(1-z^2).$

Change of the variable form x to z will now give

$$\frac{d^2y}{dx^2}-2\,y=2\,(1-z^2).$$

Here $C.\,F.=c_1\,e^{\sqrt{2}\,z}+c_2\,e^{-\sqrt{2}\,z}$

$$=c_1\,e^{\sqrt{2}\sin x}+c_2\,e^{-\sqrt{2}\sin x},$$

and $P.\,I.=\dfrac{1}{D^2-2}\,2\,(1-z^2)$

$$=-\tfrac{1}{2}\,(1+\tfrac{1}{2}\,D^2)\,\{2\,(1-z^2)\}=z^2=\sin^2 x.$$

Hence the complete solution is

$$y=c_1\,e^{\sqrt{2}\sin x}+c_2\,e^{-\sqrt{2}\sin x}+\sin^2 x.$$

Ex. 18. *Solve* : $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\,\cos\log(1+x)$

(*Agra* 1950, *Punjab* '54)

Dividing by $(1+x)^2$, the equation becomes

$$\frac{d^2y}{dx^2}+\frac{1}{1+x}\frac{dy}{dx}+\frac{1}{(1+x)^2}\,y=\frac{1}{(1+x)^2}\,4\cos\log(1+x)$$

Find z such that $\left(\dfrac{dz}{dx}\right)^2=\dfrac{1}{(1+x)^2}$ or $dz=\dfrac{dx}{1+x}$

or $z=\log(1+x)$.

$$R_1=\frac{R}{\left(\dfrac{dz}{dx}\right)^2}=\frac{\{1/(1+x)^2\}\,4\cos\log(1+x)}{1/(1+x)^2}$$

$$=4\cos\log(1+x)=4\cos z.$$

Change of the variable from x to z will now give

$$\frac{d^2y}{dz^2}+y=4\cos z.$$

Here $C.\,F.=c_1\cos(z+\alpha)=c_1\cos\{\log(1+x)+\alpha\}$,

and $P.\,I=2z\sin z=2\log(1+x)\sin\{\log(1+x)\}$

Hence solution is $y=c_1\cos\{\log(1+x)+\alpha\}$

$$+2\log(1+x)\sin\{\log(1+x)\}$$

EXERCISE VIII (C)

Solve :

1. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}\tan x+y\cos^2 x=0.$

2. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+\dfrac{a^2}{x^4}y=0.$ (*Alld.* 1942, *Agra* 1956)

3. $(x^3-x)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+n^2x^3y=0.$ (*Agra* 1960)

4. $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0$

5. $\dfrac{d^2y}{dx^2}-\cot x\dfrac{dy}{dx}-y\sin^2 x=0.$ (*Rajasthan* 1955)

6. $\sin^2 x\dfrac{d^2y}{dx^2}+\sin x\cos x\dfrac{dy}{dx}+y=0.$ (*Mysore* 1949)

7. $(1+x^2)^2\dfrac{d^2y}{dx^2}+2x(1+x^2)\dfrac{dy}{dx}+4y=0.$ (*Agra* 1961)

8. $\dfrac{d^2y}{dx^2}+(\tan x-1)^2\dfrac{dy}{dx}-n(n-1)y\sec^4 x=0.$

9. $\dfrac{d^2y}{dx^2}+(3\sin x-\cot x)\dfrac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x.$ (*Raj.* 1957)

8·5. Solution by Factorisaion of the Operator. If the left hand side of a linear equation of the second order be put in the form of $f(D)y$, then it is sometimes convenient to factorise $f(D)$ in the form $f_2(D)f_1(D)$, such that when $f_1(D)$ operates upon y and then $f_2(D)$ operates upon the result of operation, we get the same result as if $f(D)$ operates upon y.

Of course it is only in special cases that the operator can be factorised. It should be noted that in most cases these factors are not commutative as they contain variables and therefore *must be written in the right order.*

The following examples will illustrate the method.

Ex. 19. *Solve :* $x\dfrac{d^2y}{dx^2}+(x-1)\dfrac{dy}{dx}-y=x^2.$

(*Agra* 1950, *Luck.* 1951)

The equation may be written as

$$\{xD^2+(x-1)D-1\}y=x^2,$$

where D stands for the operator $\dfrac{d}{dx}$.

Using symbolic factors it becomes

$$(xD-1)(D+1)\,y=x^2. \qquad (1)$$

[We can not write it as $(D+1)(xD-1)\,y=x^2$ for

$$(D+1)(xD-1)\,y=[\,xD^2+(x+1)\,D-1\,]\,y.$$

Let $(D+1)\,y=v$, then (1) becomes

$$(xD-1)\,v=x^2 \quad \text{or} \quad \frac{dv}{dx}-\frac{1}{x}\,v=x,$$

which on integration gives $v=x^2+c_1 x$.

Substituting this value of v in $(D+1)\,y=v$, it becomes

$$(D+1)\,y=x^2+c_1 x$$

or $$\frac{dy}{dx}+y=x^2+c_1 x$$

which on integration gives

$$y=x^2+c_1(x-1)+c_2 e^{-x}.$$

Ex. 20. *Solve* : $3x^2\dfrac{d^2y}{dx^2}+(2+6x-6x^2)\dfrac{dy}{dx}-4y=0.$

The equation may be written as

$$\{\,3x^2 D^2+(2+6x-6x^2)\,D-4\,\}\,y=0.$$

Factorising we can write it as

$$(D-2)(3x^2 D+2)\,y=0 \qquad (1)$$

[We cannot write it as $(3x^2 D+2)(D-2)\,y=0$ for

$$(3x^2 D+2)(D-2)\,y=(3x^2 D^2-6x^2 D+2D-4)\,y\;]$$

Let $(3x^2 D+2)\,y=v$. (2)

Therefore (1) becomes

$$(D-2)\,v=0 \qquad \therefore\; v=c_1 e^{2x}.$$

Substituting this value of v in (2)

$$3x^2\frac{dy}{dx}+2y=c_1 e^{2x} \quad \text{or} \quad \frac{dy}{dx}+\frac{2}{3x^2}\,y=\frac{c_1}{3x^2}\,e^{2x},$$

which on integration gives

$$y\,e^{-2/3x}=c_2+c_1\int e^{-2/3x}\,e^{2x}.\,\frac{1}{3x^2}\,dx$$

$$\therefore\; y=c_2 e^{2/3x}+c_1 e^{2/3x}\int e^{2x-2/3x}.\,\frac{1}{3x^2}\,dx.$$

EXERCISE VIII (D)

Solve :

1. $3x^2\dfrac{d^2y}{dx^2}+(2-6x^2)\dfrac{dy}{dx}-4y=0.$

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2. $x\dfrac{d^2y}{dx^2}+(x-2)\dfrac{dy}{dx}-2y=x^3.$ (*I. A. S.* 1957)

3. $x^2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-(1+x^2)\,y=e^{-x}.$

4. $(x+2)\dfrac{d^2y}{dx^2}-(2x+5)\dfrac{dy}{dx}+2y=(x+1)\,e^x.$

8·6. Method of variation of parameters. Now we shall explain this method of finding the complete primitive when the complementary function is known.

Let us consider the case of the general linear equation

$$y_2+Py_1+Qy=R.$$

Let $y=Au+Bv$ be the complementary function where A and B are constants and u and v are functions of x.

Let us assume that

$$y=Au+Bv$$

is the complete primitive of (1) where A and B are no longer constants but functions of x to be so chosen that the equation (1) will be satisfied.

Differentiating (2) we get

$$y_1=Au_1+Bv_1+uA_1+vB_1$$

Now the two unknown functions A and B can be made to satisfy two conditions. So far only one relation between them is known. [This can be obtained by substituting (2) in (1)]. Thus we may choose the other relation according to our convenience. Let it be given by

$$uA_1+vB_1=0, \qquad \text{......(3)}$$

so that $\quad y_1=Au_1+Bv_1.$ ......(4)

Differentiating (4)

$$y_2=Au_2+Bv_2+u_1\,A_1+v_1\,B_1.$$

Substituting for y, y_1 and y_2 in (1).

The coefficient of A will be u_2+Pu_1+Qu which is zero by hypothesis.

Similarly the coefficient of B is zero.

Therefore (1) will reduce to

$$u_1\,A_1+v_1\,B_1=R.$$

From (3) and (6) we get

$$\frac{A_1}{v}=\frac{B_1}{-u}=\frac{-R}{u\,v_1-u_1\,v}.$$

Thus $A_1 = \dfrac{-v\ R}{u\ v_1 - u_1\ v}$ and $B_1 = \dfrac{u\ R}{u\ v_1 - u_1\ v}$

We then get A and B by integration.

Let $A = f(x) + a$

$B = \phi\ (x) + b$

$\therefore\quad y = u\, f\ (x) + v\ \phi\ (x) + au + bv,$

The above method can be extended to linear equations of any order.

Let us consider

$$\frac{d^3y}{dx^3} + P\ \frac{d^2y}{dx^2} + Q\,\frac{dy}{dx} + Ry = S. \qquad \ldots\ldots(4)$$

Let $y = u$, $y = v$, $y = w$ be the solutions of the given equation when $S = 0$.

Let $y = Au + Bv + Cw,$ $\qquad \ldots\ldots(5)$

where A, B, and C are no longer constants but functions of x to be so chosen that (5) may satisfy (4). Differentiating

$$y_1 = Au_1 + Bv_1 + Cw_1 + uA_1 + vB_1 + wC_1.$$

We can now choose two relations according to our convenience between three unknown functions A, B, C as they can be made to satisfy three conditions, only one of which is given.

Let us assume as one of the relations

$$uA_1 + vB_1 + wC_1 = 0. \qquad \ldots\ldots(6)$$

$$\therefore\quad y_1 = Au_1 + Bv_1 + Cw_1 \qquad \ldots\ldots(7)$$

Differentiating $y_2 = Au_2 + Bu_2 + Cw_2$, taking for second relation

$$u_1\ A_1 + v_1\ B_1 + w_1\ C_1 = 0 \qquad \ldots\ldots(8)$$

Also $y_3 = Au_3 + Bv_3 + Cw_3 + u_2\ A_1 + v_2\ B_1 + w_2\ C_1$

Substituting for y, y_1, y_2 and y_3 in (5) we get

$$u_2\ A_1 + v_2\ B_1 + w_2\ C_1 = S \qquad \ldots\ldots(9)$$

Solving (6), (8) and (9) we get A_1, B_1 and C_1 which by integration will give A, B and C.

As the solution is obtained by varying the arbitrary constants of the complementary function the above method is known as variation of parameters.

We have previously seen that a linear equation of the second order may be solved completely when one integral belonging to the complementary function has been found. There is, no doubt, the superior efficiency of this method over that of the variation of parameters, as latter requires a complete knowledge of the complementary function instead of only one part of it. Besides that if this method is applied to equations of the third or higher orders, it requires too much labour to solve a number of simultaneous equations.

Ex. 21. *Solve :* $\dfrac{d^2y}{dx^2}+a^2y=sec\ ax.$

(*Nagpur* 1961, *Agra* 1945, 1952, 1961 *Alld.* 1949, *Raj.* '50, 52)

Particular solutions of the equation when the right-hand member is zero are $u=\cos ax$ and $v=\sin ax$.

Let the primitive be

$$y=A\cos ax+B\sin ax$$

Two equations to connect A and B are

$$uA_1+vB_1=0 \text{ and } u_1 A_1+v_1 B_1=R$$

Therefore $\qquad A_1\cos ax+B_1\sin ax=0 \qquad$ ...(1)

and $\qquad -A_1 a\sin ax+B_1 a\cos ax=\sec ax \qquad$ ...(2)

Solving (1) and (2)

$$A_1=-(1/a)\tan ax \text{ and } B_1=(1/a)$$

Therefore $\qquad A=C+(1/a^2)\log\cos ax$ and $B=(x/a)+D$

The primitive is therefore

$$y_1=C\cos ax+(1/a^2)\cos ax.\log(\cos ax)+D\sin ax+(x/a)\sin ax.$$

Ex. 22. *Solve :* $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^2e^x.$

(*Agra* 1935, 1943, 1960 ; *I. A. S.* 1954)

Arranging in the standard form, this is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x.$$

Particular solutions of the equation when the right-hand member is zero are $u=x$ and $v=x^{-1}$.

Let the primitive be

$$y=Ax+Bx^{-1}.$$

Two equations to connect A and B are

$$A_1x+B_1x^{-1}=0 \qquad ...(1)$$

and $$A_1-\frac{1}{x^2}B_1=e^x \qquad ...(2)$$

Solving (1) and (2) $A_1=\frac{1}{2}e^x$, whence $A=\frac{1}{2}e^x+a$

and $\qquad B_1=-\frac{1}{2}e^x.x^2$, whence $B=-\frac{1}{2}e^x x^2+x e^x-e^x+b.$

Hence the complete primitive is

$$y=\tfrac{1}{2}x e^x+a x-\tfrac{1}{2}e^x.x+e^x-e^x x^{-1}+b x^{-1}.$$
$$=a x+b x^{-1}+e^x-e^x x^{-1}.$$

Ex. 23. *Solve :* $x^2\dfrac{d^2y}{dx^2}-2x(1+x)\dfrac{dy}{dx}+2(1+x)y=x^3,$

the integrals in the complementary function are $y=x,\ y=x\,e^{2x}.$

(*Jaipur* 1955, *Agra* 1957)

Rearranging this as

$$\frac{d^2y}{dx^2}-\frac{2}{x}(1+x)\frac{dy}{dx}+\frac{2}{x^2}(1+x)\,y=x.$$

Let the primitive be

$$y=Ax+Bx\,e^{2x}.$$

Two equations to connect A and B are

$$A_1\,x+x\,e^{2x}\,B_1=0,$$

and $$A_1+(1+2x)\,e^{2x}\,B_1=x.$$

Solving these two equations

$$A_1=-\tfrac{1}{2},\quad \text{whence}\quad A=-\tfrac{1}{2}x+a\,;$$

and $$B_1=\tfrac{1}{2}e^{-2x},\ \text{whence}\quad B=-\tfrac{1}{4}\,e^{-2x}+b.$$

Hence the primitive is

$$\begin{aligned} y&=Ax+Bx\,e^{2x}\\ &=(-\tfrac{1}{2}x+a)\,x+(-\tfrac{1}{4}\,e^{-2x}+b)\,x\,e^{2x}\end{aligned}$$

or $$\begin{aligned}2y&=(-x+2a-\tfrac{1}{2}+2b\,e^{2x})\,x\\ &=(-x+C+D\,e^{2x})\,x.\end{aligned}$$

Ex. 24. *Solve :* $\dfrac{d^2y}{dx^2}+(1-\cot x)\dfrac{dy}{dx}-y\cot x=\sin^2 x.$

(*Jaipur* 1956, 1958)

(v. imp)

Let the left hand side equated to zero be solved.

Obviously $y=e^{-x}$ is a solution of it.

Let $y=v\,e^{-x}$. Then the is equation is reduced to

$$\frac{d^2v}{dx^2}-(1+\cot x)\,\frac{dv}{dx}=0$$

which on integration gives

$$\begin{aligned} c_1\,v&=-\tfrac{1}{2}e^{x}\,(\cos x-\sin x)+c_2\\ y&=vy_1=A\,(\cos x-\sin x)+Be^{-x}\end{aligned}$$

$\therefore$ The C F. of the given equation is

$$y=A\,(\cos x-\sin x)+Be^{-x}$$

Two equations to connect A and B are

$$A_1\,(\cos x-\sin x)+B_1\,e^{-x}=0 \qquad \text{...(1)}$$

and $$A_1\,(-\sin x-\cos x)-B_1\,e^{-x}=\sin^2 x \qquad \text{...(2)}$$

Solving (1) and (2) $A_1=-\tfrac{1}{2}\sin x$ whence $A=\tfrac{1}{2}\cos x+c_1$

and $$B_1=\tfrac{1}{4}\,e^{x}\sin 2x-\tfrac{1}{4}\,e^{x}\,(1-\cos 2x)$$

whence $B=-\tfrac{1}{20}\,e^{x}\,(2\cos 2x-\sin 2x)-\tfrac{1}{4}\,e^{x}$
$\qquad -\tfrac{1}{20}\,e^{x}\,(-2\sin 2x-\cos 2x)+c_2$

$\therefore$ Complete solution is

$$\begin{aligned} y&=A\,(\cos x-\sin x)+B\,e^{-x}\\ &=c_1\,(\cos x-\sin x)-\tfrac{1}{10}\,(\sin^2 2x-2\cos 2x)+c_2\,e^{-x}.\end{aligned}$$

Ex 25. *Solve* : $y_3-6y_2+11y_1-6y=e^{2x}$. (*Agra* **1939, 1962**)

Particular solutions of the equation when the right hand member is zero are $u=e^x$, $v=e^{2x}$, $w=e^{3x}$.

Let the primitive be

$$y=A\,e^x+B\,e^{2x}+C\,e^{3x}.$$

Three equations to connect A, B and C are

$$e^x A_1+e^{2x} B_1+e^{3x} C_1=0,$$
$$e^x A_1+2e^{2x} B_1+3\,e^{3x} C_1=0 \qquad \ldots(2)$$
$$e^x A_1+4\,e^{2x} B_1+9\,e^{2x} C_1=e^{2x}. \qquad \ldots(3)$$

From (1) and (2)

$$\frac{A_1}{e^{5x}}=\frac{B_1}{-2e^{4x}}=\frac{C_1}{e^{3x}}=\lambda \text{ (say)}$$

Substituting the values of A_1, B_1, C_1 in (3)

$$e^{2x}=\lambda\ (e^{6x}-8e^{6x}+9e^{6x})=\lambda\ 2e^{6x}$$

Hence $\lambda=\frac{1}{2}\,e^{-4x}$.

$$A_1=\tfrac{1}{2}\,e^x,\ B_1=-1,\ C_1=\tfrac{1}{2}e^{-x}$$

Integrating $A=\frac{1}{2}\,e^x+a$, $B=-x+b$, $C=-\frac{1}{2}\,e^{-x}+c$

The complete primitive, therefore is

$$y=\tfrac{1}{2}\,e^{2x}+a\,e^x-x\,e^{2x}+be^{2x}-\tfrac{1}{2}e^{2x}+ce^{3x}$$
$$=a\,e^x+b\,e^{2x}+c\,e^{3x}-x\,e^{2x}.$$

EXERCISE VIII (E)

Solve the following by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2}+y=x.$ 2. $\dfrac{d^2y}{dx^2}+y=\operatorname{cosec} x.$ (*Agra* **1949**)

3. $y_2+4y=4\tan 2x.$ (*Agra* **1949, 1954**)

4. $(1-x)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=(1-x)^2.$

[Complementary function is $y=A\,e^x+Bx$].

5. $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$ (*Agra* **1950**)

[Complementary function is $y=A\,e^x+B\,e^{-x}$].

6. $(1-x^2)\,y_2-4xy_1-(1+x^2)\,y=x.$ (*Jaipur* **1953**, *Agra* **1940**)

7. $x^2\,y_2-2x\,(1+x)\,y_1+2\,(1+x)\,y=-4x^3.$ (*Agra* **1936**)

8. $x\,y_1-y=(x-1)\,(y_2-x+1).$ (*Agra* **1945**, *Jaipur* **1949**)

8·7. Synopsis. If the methods given in the previous chapters fail to solve a linear equation of second order and we have to resort to methods given in this chapter the following procedure is suggested :

1. First of all put the equation in the form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R. \tag{A}$$

(*a*) Then try to find by inspection or by easy calculation an integral belonging to the complementary function of the given equation ; then use Art. 8·2.

(*b*) If it is not possible to proceed as in (*a*), find the value of $Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2$. If it is a constant or a constant divided by x^2 the Normal form of the equation will be easily integrable and hence also the original equation.

(*c*) If the methods given in Art. 8·2 and 8·3 are not applicable the method of change of independent variable may be tried. It will be of advantage to remember the values of P_1, Q_1, R_1 as given in Art. 8·4 especially the value of z which makes $P_1=0$.

2. *In a few special cases* the method of factorisation will easily effect the integration. It must be carefully noted that factors are not commutative on account of the presence of variables. In this case we proceed straightaway without putting the equation in the standard form (*A*).

3. Method of variation of parameters is to be used if instructed to do so.

Miscellaneous Exercise on Chapter VIII

1. Solve $x^2 y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0.$ (*Agra* 1938)

[It is not a linear equation but can be solved with the help of Art. 8·1 ; $y=x$ is a solution of the equation. Substituting

$$y=vx,\ \frac{dy}{dx}=v+x\frac{dv}{dx}\ \text{and}\ \frac{d^2y}{dx^2}=2\frac{dv}{dx}+x\frac{d^2v}{dx^2}$$

the equation becomes $2xv\frac{d^2v}{dx^2}+2x\left(\frac{dv}{dx}\right)^2+4v\frac{dv}{dx}=0$

It is an exact equation, hence integrating $2xv\frac{dv}{dx}+v^2=c_1$

we get $xv^2=c_1x+c_2$. This is still exact $\therefore\ y^2=c_1x^2+c_2x$].

2. Solve the equations :

(*i*) $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$ (*Agra* '56, '59)

(*ii*) $(a+x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0$

(*iii*) $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(n^2+\dfrac{2}{x^2}\right)y=0$

(*iv*) $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+1)\,y=x^3+3x$ (*Luck*. 1955)

(*v*) $(a^2-x^2)\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x}\dfrac{dy}{dx}+\dfrac{x^2}{a}\,y=0$ (*Agra* 1957)

(*vi*) $x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}+n^2\,y=0$

(*vii*) $(1-x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+\dfrac{a^2y}{(1-x^2)}=0$

(*viii*) $(2x-1)\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+(3-2x)\,y=2e^x.$ (*Luck*. 1955)

(*ix*) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=8x^3.$

(*x*) $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+5)\,y=x\,e^{-\frac{1}{2}x^2}$ *Luck*. '49, '54)

3. Verify that $(1-x^2)$ is a particular solution of the equation

$$x\,(1-x^2)^2\,\frac{d^2y}{dx^2}+(1-x^2)\,(1+3x^2)\,\frac{dy}{dx}+4x\,(1+x^2)\,y=0$$

and solve it completely.

By the variation of parameters, solve completely the equation obtained by writing $(1-x^2)^3$ instead of zero on the right-hand-side of the given equation (*Luck*. '56)

4. Assuming that the primitive of

$$\frac{d^2y}{dx^2}+\left(1-\frac{2}{x^2}\right)y=0$$

is of the form $y=u+\dfrac{v}{x}$, prove that it is given by

$$u=A\sin(x+\alpha),\ v=A\cos(x+\alpha)$$

Obtain the primitive of

$$\frac{d^2y}{dx^2}+\left(1-\frac{2}{x^3}\right)y=x^2.$$ (*Agra* 1938)

5. Solve the following differential equations :

(i) $(x^3-2x^2)\dfrac{d^2y}{dx^2}+2x^2\dfrac{dy}{dx}-12(x-2)\,y=0.$

(*Nagpur* 1953)

(ii) $x\dfrac{d^2y}{dx^2}-2(x+1)\dfrac{dy}{dx}+(x+2)\,y=(x-2)\,e^{2x}.$

(iii) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$, given that $x+\dfrac{1}{x}$ is an integral.

(iv) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-9\,y=0$, given that $y=x^3$ is a solution.

(v) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-a^2\,y=0$, of which $y=ce^{a\sin^{-1}x}$ is an integral.

(*Agra* 1938)

(vi) $x\dfrac{d^2y}{dx^2}(x\cos x-2\sin x)+(x^2+2)\dfrac{dy}{dx}\sin x$

$-2y\,(x\sin x+\cos x)=0$, given that $y=x^2$ is a solution.

(vii) $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+4x^3\,y=x^5.$

(viii) $(x^2-1)\dfrac{d^2y}{dx^2}-(4x^2-3x-5)\dfrac{dy}{dx}+(4x^2-6x-5)\,y$

$=e^{2x}$, given that $y=1$, $\dfrac{dy}{dx}=2$ when $x=0$.

(ix) $(x^2-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=m^2y.$

(x) $\dfrac{d^2y}{dx^2}+\left(1-\dfrac{1}{x}\right)\dfrac{dy}{dx}+4x^2\,y\,e^{-2x}=4\,(x^2+x^3)\,e^{-3x}.$

(*Agra* 1937)

(xi) $x\dfrac{d^2y}{dx^2}+(x^2+1)\dfrac{dy}{dx}+2xy=2x$, given that $y=2$ and $\dfrac{dy}{dx}=0$ when $x=0$.

6. Solve the following by the method of variation of parameters :

(*i*) $(x+2)\,y_2-(2x+5)\,y_1+2y=(x+1)\,e^x$.

(*Raj* 1959 *Nag.* 1958, *Agra* 1947, '52, '59)

(*ii*) $(1-x^2)\,y_2+xy_1-y=x\,(1-x^2)^{3/2}$ (*Jaipur* 1954)

7. Show that if the two equations

$$y_2+Py_1+Qy=0,$$

and $$z_2+pz_1+qz=0$$

reduce to the same Normal Form, they may be transformed into each other by the relation

$$ye^{\frac{1}{2}\int P dx}=ze^{\frac{1}{2}\int p dx},$$

i.e. the condition of equivalence is that the *Invariant I* should be the same.

8. If u and su are any two solutions of

$$v_2+Iv=0, \qquad (1)$$

Prove that $\dfrac{s_2}{s_1}=-2\dfrac{u_1}{u}$ (2)

and hence that $\dfrac{s_3}{s_1}-\dfrac{3}{2}\left(\dfrac{s_2}{s_1}\right)^2=2I$ (3)

From (2) show that if s is any solution of (3), $s_1^{-1/2}$ and $ss_1^{-1/2}$ are solutions of (1).

9. Calculate the Invariant I of the equation

$$x^2\,y_2-(x^2+2x)\,y_1+(x+2)\,y=0.$$

Taking s as the quotient of the two solutions xe^x and x, verify that the relation (3) of *Q*. 8. is satisfied and that $s_1^{-1/2}$ and $ss_1^{-1/2}$ are solutions of the Normal Form of the original equation.

10. If u and v are two solutions of

$$y_2+Py_1+Qy=0,$$

prove that $uv_2-vu_2+P\,(uv_1-vu_1)=0,$

and hence show that $uv_1-vu_1=ae^{-\int Pdx}$

CHAPTER IX

SIMULTANEOUS EQUATIONS

9·1. We have considered so far only those differential equations which contained two variables. Now we proceed to consider a few forms containing more than two variables. In such equations the simplest form is that which contains only one independent variable of which all other variables are functions. In this system the number of equations to connect these variables is always equal to the number of dependent variables.

9·2. Simultaneous equations with constant coefficients.

There are two methods to solve such equations : Let x, y be dependent variables and t be the independent variable.

Symbolic Method—To solve the linear equations.

$$f_1(D)\,x+F_1(D)\,y=T_1, \tag{1}$$
$$f_2(D)\,x+F_2(D)\,y=T_2, \tag{2}$$

where T_1, T_2, are functions of the independent variable t, D stands for the operator $\frac{d}{dt}$ and f_1, f_2, F_1, F_2, are rational integral functions with constant coefficients.

Operating on both the sides of equation (1) with $F_2(D)$ and both the sides of equation (2) with $F_1(D)$, we get

$$F_2(D)\,f_1(D)\,x+F_2(D)\,F_1(D)\,y=F_2(D)\,T_1 \tag{3}$$
$$F_1(D)\,f_2(D)\,x+F_1(D)\,F_2(D)\,y=F_1(D)\,T_2. \tag{4}$$

Since the functions F_1 and F_2 have only constants in their coefficients, it follows that

$$F_2(D)\,F_1(D)\,y=F_1(D)\,F_2(D)\,y$$

Hence subtracting (4) from (3)

$$\{F_2(D)\,f_1(D)-F_1(D)\,f_2(D)\}\,x=F_2(D)\,T_1-F_1(D)\,T_2.$$

This can be integrated to give the value of x by a method applicable to an ordinary single equation. Now the value of y can be obtained by substituting the value of x in either of the two given equations. If however, y is determined by an independent elimination, as the case of x, the values of x and y will have to be substituted in equation (1) or (2) and the arbitrary constants in, say y, adjusted so that the equation may be satisfied.

The following examples will make the process clear.

Ex. 1. *Solve* : $\dfrac{dx}{dt}-7x+y=0,\ \dfrac{dy}{dt}-2x-5y=0.$

(*Agra* 1947, 1958)

Writing in the symbolic form, we have

$$(D-7)\,x+y=0 \qquad (1)$$
$$-2x+(D-5)\,y=0 \qquad (2)$$

Operating both the sides of (1) by $(D-5)$ we get

$$(D-5)\,(D-7)\,x+(D-5)\,y=0. \qquad (3)$$

Subtracting (2) from (3)

$$\{\,(D-5)\,(D-7)+2\,\}\,x=0$$
or $$(D^2-12\,D+37)\,x=0$$

and integrating, $x=e^{6t}\,(A\cos t+B\sin t)$ (4)

Differentiating (4), $\dfrac{dx}{dt}=6\,e^{6t}\,(A\cos t+B\sin t)$

$$+e^{6t}\,(-A\sin t+B\cos t) \qquad (5)$$

Substituting these values in the first equation

$$y=7x-Dx=e^{6t}\,\{\,(A-B)\cos t+(A+B)\sin t\,\}.$$

Aliter. Having calculated x, we may calculate the value of y, as follows :

Multiplying (1) by 2 and operating (2) by $(D-7)$ we get

$$2\,(D-7)\,x+2y=0,\ -2\,(D-7)\,x+(D-7)\,(D-5)\,y=0.$$

$\therefore$ Eliminating x, we get

$[\,(D-7)\,(D-5)+2\,]\,y=0$ *i.e.* $(D^2-12D+37)\,y=0$

$\therefore\ y=e^{6t}\,(c_1\cos t+c_2\sin t)$

Substituting the value of y in (1) along with the value of x,

$$e^{6t}\,[\,(c_1-A+B)\cos t+(c_2-A-B)\sin t\,]=0$$

$\therefore\ c_1=A-B,\ c_2=A+B$

$\therefore\ y=(A-B)\cos t+(A+B)\sin t.$

Ex. 2. *Solve the simultaneous equations*

$$\frac{dx}{dt}=ax+by,\ \frac{dy}{dt}=a'x+b'y.$$ (*Agra* 1961)

These equations can be expressed in symbolic form as

$$(D-a)\,x-by=0 \qquad ...(1)$$
$$-a'x+(D-b')\,y=0 \qquad ...(2)$$

In order to eliminate x we multiply (1) by a' and operate (2) by $D-a$.

$$a'\,(D-a)\,x-a'by=0$$
$$a'\,(D-a)\,x+(D-a)\,(D-b')\,y=0.$$

Subtracting we get

$$\{ (D-a)\ (D-b')+a'b \}\ y=0$$

or
$$\{ D^2-(a+b')\ D+ab'+a'b \}\ y=0$$

or
$$D=\frac{(a+b') \pm \sqrt{(a+b')^2-4ab'-4a'b}}{2}$$

$$=\tfrac{1}{2} \{ (a+b') \pm \sqrt{(a-b')^2-4a'b} \}$$

$=m$ with positive sign and n with negative sign (say).

$\therefore \quad y=c_1 e^{mt}+c_2 e^{nt}$, where m and n have the above values

Now $\dfrac{dy}{dt}=c_1 me^{mt}+c_2 ne^{nt}$.

Substituting the values of y and $\dfrac{dy}{dt}$ in the second eq^n we get

$$\frac{dy}{dt}-a'x-b'y=0$$

or
$$c_1 me^{mt}+c_2 ne^{nt}-b'\ (c_1 e^{mt}+c_2 e^{nt})=a'x$$

or
$$(m-b')\ c_1 e^{mt}+(n-b')\ e^{nt}=a'x$$

$$\therefore \quad x=(1/a')\ [\ (m-b')\ c_1 e^{mt}+(n-b')\ e^{nt}\ ].$$

Thus we get the values of x and y in terms of m and n

where $\quad m=\tfrac{1}{2} \{ (a+b')+\sqrt{(a-b')^2-4a'b} \}$

and $\quad n=\tfrac{1}{2} \{ (a+b')-\sqrt{(a-b')^2-4a'b} \}$

Method of Differentiation. Let the two given equations connect t with the four quantities x, y, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$. Differentiating them with respect to t, four equations containing x, y, $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, $\dfrac{d^2x}{dt^2}$ and $\dfrac{d^2y}{dt^2}$ are obtained and from these four y, $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$ are eliminated. In this way an equation of the second order, in which x is the dependent and t the independent variable is obtained. Solved example 3 given below will illustrate the method.

Ex. 3. *Solve:* $\quad \dfrac{dx}{dt}+5x+y=e^t, \qquad \ldots(1)$

$$\frac{dy}{dt}-x+3y=e^{2t}. \qquad \ldots(2)$$

Differentiating equation (1) we have

$$\frac{d^2x}{dt^2}+5\ \frac{dx}{dt}+\frac{dy}{dt}=e^t. \qquad ...(3)$$

Eliminating $\frac{dy}{dt}$ and y from (3) with the help of (1) and (2) we have

$$\frac{d^2x}{dt^2}+8\ \frac{dx}{dt}+16x=4\ e^t-e^{2t}.$$

The $C.\ F.$ is $(c_1+c_2\ t)\ e^{-4t}$ and the

$$P.\ I.=\frac{1}{(D+4)^2}\ 4\ e^t-\frac{1}{(D+4)^2}\ e^{2t}$$

$$=\tfrac{4}{25}\ e^t-\tfrac{1}{36}\ e^{2t}.$$

Therefore $x=(c_1+c_2\ t)\ e^{-4t}+\frac{4}{25}\ e^t-\frac{1}{36}\ e^{2t}$,

and $\frac{dx}{dt}=-4\ (c_1+c_2\ t)\ e^{-4t}+c_2\ e^{-4t}+\frac{4}{25}\ e^t-\frac{1}{18}\ e^{2t}$.

Substituting these values in (1)

$$y=-(c_1+c_2+c_2\ t)\ e^{-4t}+\tfrac{7}{36}\ e^{2t}+\tfrac{1}{25}\ e^t.$$

Ex. 4. *Solve* : $\frac{d^2x}{dt^2}-3x-4y=0,$

$$\frac{d^2y}{dt^2}+x+y=0. \qquad (\textit{Agra}\ 1956,\ 1962)$$

Writing in the symbolic form $\left(\text{ i.e. } D \text{ for } \frac{d}{dt}\right)$, we have

$$(D^2-3)\ x-4y=0, \qquad ...(1)$$
$$x+(D^2+1)\ y=0. \qquad ...(2)$$

Operating both the sides of (1) by (D^2+1) and multiplying both the sides of (2) by 4 we get

$$(D^2+1)\ (D^2-3)\ x-4\ (D^2+1)\ y=0 \qquad ...(3)$$
$$4x+4\ (D^2+1)\ y=0 \qquad ...(4)$$

Adding (3) and (4) we get

$$[(D^2+1)\ (D^2-3)+4]\ x=0$$

i.e. $\qquad (D^2-1)^2\ x=0$

or $\qquad (D-1)^2\ (D+1)^2\ x=0.$

$\therefore\quad x=(c_1+c_2\ t)\ e^t+(c_3+c_4\ t)\ e^{-t}$

Again $\dfrac{dx}{dt}=(c_1+c_2+c_2\,t)\,e^t-(c_2+c_4\,t-c_4)\,e^{-t}$

and $\dfrac{d^2x}{dt^2}=(c_1+2c_2+c_2\,t)\,e^t+(c_3+c_4\,t-2c_4)\,e^{-t}.$

Substituting these values in the first equation

$$4y=\frac{d^2y}{dt^2}-3x$$

$$=(c_1+2c_2+c_2\,t-3c_1-3c_2\,t)\,e^t+(c_3+c_4\,t-2c_4-3c_3-3c_4\,t)\,e^{-t}$$

$$=(2c_2-2c_1-2c_2\,t)\,e^t-(2c_3+2c_4\,t+2c_4)\,e^{-t}$$

or $y=\frac{1}{2}\,[(c_2-c_1-c_2\,t)\,e^t-(c_3+c_4\,t+c_4)\,e^{-t}.$

Ex. 5. *Solve* : $\dfrac{d^2x}{dt^2}+m^2y=0$ and $\dfrac{d^2y}{dt^2}-m^2\,x=0.$

(*Agra* 1953)

Writing in the symbolic form, we have

$$D^2x+m^2\,y=0 \qquad \ldots\ldots(1)$$

$$-m^2\,x+D^2\,y=0. \qquad \ldots\ldots(2)$$

Operating both the sides of (1) by D^2 and multiplying both sides of (2) by m^2, we get

$$D^4\,x+m^2\,D^2\,y=0 \qquad \ldots\ldots(3)$$

$$-m^4\,x+m^2\,D^2\,y=0. \qquad \ldots\ldots(4)$$

Subtracting (4) from (3)

$$(D^4+m^4)\,x=0$$

or $(D^2-\sqrt{2}Dm+m^2)\,(D^2+\sqrt{2}Dm+m^2)\,x=0.$

Hence $x=e^{mt/\sqrt{2}}\left(c_1\cos\dfrac{mt}{\sqrt{2}}+c_2\sin\dfrac{mt}{\sqrt{2}}\right)$

$$+e^{-mt/\sqrt{2}}\left(c_3\cos\frac{mt}{\sqrt{2}}+c_4\sin\frac{mt}{\sqrt{2}}\right)$$

Differentiating twice and substituting the value of x in (1),

$$y=e^{mt/\sqrt{2}}\left(c_1\sin\frac{mt}{\sqrt{2}}-c_2\cos\frac{mt}{\sqrt{2}}\right)$$

$$+e^{-mt/\sqrt{2}}\left(c_4\cos\frac{mt}{\sqrt{2}}-c_3\sin\frac{mt}{\sqrt{2}}\right).$$

Ex. 6. *Solve* : $4\dfrac{dx}{dt}+9\dfrac{dy}{dt}+11x+31y=e^t,$

$$3\frac{dx}{dt}+7\frac{dy}{dt}+8x+24y=e^{2t}$$

Writing in the symbolic form,

$$(4D+11)\,x+(9D+31)\,y=e^t, \qquad \ldots\ldots(1)$$
$$(3D+8)\,x+(7D+24)\,y=e^{2t}. \qquad \ldots\ldots(2)$$

Operating both the sides of (1) by $(7D+24)$ and both the sides of (2) by $(9D+31)$, we get

$$(7D+24)\,(4D+11)\,x+(7D+24)\,(9D+31)\,y=(7D+24)\,e^t \quad (3)$$
$$(9D+31)\,(3D+8)\,x+(9D+31)\,(7D+24)\,y=(9D+31)\,e^{2t} \quad (4)$$

Subtracting (4) from (3)

$$(D^2+8D+16)\,x=7\,e^t+24\,e^t+18\,e^{2t}+31\,e^{2t}$$

or $\qquad (D+4)^2\,x=31\,e^t+49\,e^{2t}$

$\therefore\quad C.\,F.=(c_1+c_2\,t)\,e^{-4t}$

and $\quad P.\,I.=\dfrac{1}{(D+4)^2}\,31\,e^t+\dfrac{1}{(D+4)^2}\,49\,e^{2t}$

$$=\tfrac{31}{25}e^t+\tfrac{49}{36}e^{2t}$$

Hence $\quad x=(c_1+c_2\,t)\,e^{-4t}+\tfrac{31}{25}e^t+\tfrac{49}{36}e^{2t}.$

Similarly the value of y can be obtained.

Ex. 7. *Solve* : $t\,dx=(t-2x)\,dt,\ t\,dy=(tx+ty+2x-t)\,dt.$

Adding these two equations (*Bombay* 1961)

$$t\,dx+t\,dy=(tx+ty)\,dt$$

or $\quad dx+dy=(x+y)\,dt$

or $\quad \dfrac{dx+dy}{x+y}=dt.$

Integrating $\log\,(x+y)=t+\text{constant}$ or $x+y=be^t.$ ......(3)

Equation (1) gives $t^2\dfrac{dx}{dt}+2t\,x=t^2$

or $\quad t^2\,x=\tfrac{1}{3}\,t^3+c$

$\therefore\quad x=\tfrac{1}{3}\,t+ct^{-2}.$ ......(4)

Equations (3) and (4) give the values of x and y.

Ex. 8. *Solve* : $\dfrac{dx}{dt}+\dfrac{2}{t}\,(x-y)=1,$ ......(1)

$$\frac{dy}{dt}+\frac{1}{t}\,(x+5y)=t. \qquad \ldots\ldots(2)$$

(*Lucknow* 1951)

Differentiating (1) with respect to t we get

$$t\frac{d^2x}{dt^2}+3\frac{dx}{dt}-2\frac{dy}{dt}=1. \qquad \ldots\ldots(3)$$

Eliminating y and $\frac{dy}{dt}$ in (1), (2) and (3) we easily get

$$t\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12\frac{x}{t}=2t+6$$

or $$t^2\frac{d^2x}{dt^2}+8t\frac{dx}{dt}+12x=2t^2+6t.$$

This is a homogeneous linear equation. Putting

$t=e^z$, $\frac{d}{dz}\equiv D$, we get

$$[D(D-1)+8D+12]\,x=2e^{2z}+6e^z$$

or $$(D^2+7D+12)\,x=2e^{2z}+6e^z$$

$C.F.$ is $Ae^{-4z}+Be^{-3z}$ i.e. $At^{-4}+Bt^{-3}$.

For $P.I.$, $x=\frac{1}{(D+4)(D+3)}(2e^{2z}+6e^z)$ where $D\equiv\frac{d}{dz}$, $t=e^z$

$$=e^{2z}/15+3e^z/10=t^2/15+3t/10$$

$$\therefore\quad x=At^{-4}+Bt^{-3}+t^2/15+3t/10$$

Substituting these values in (1), we get

$$y=-At^{-4}-\tfrac{1}{2}Bt^{-3}+2t^2/15-t/20.$$

Ex. 9. *Solve* : $$\frac{dx}{dt}-ny+mz=0, \qquad \ldots\ldots(1)$$

$$nx+\frac{dy}{dt}-lz=0, \qquad \ldots\ldots(2)$$

$$-mx+ly+\frac{dz}{dt}=0. \qquad \ldots\ldots(3)$$

We have

$Dx-ny+mz=0$, $nx+Dy-lz=0$ and $-mx+ly+Dz=0$.

Solving the first two we get

$$x=\frac{nl-mD}{D^2+n^2}\cdot z \text{ and } y=\frac{mn+lD}{D^2+n^2}\cdot z$$

Substituting these in the third, we get

$$\frac{d^3z}{dt^3}+(l^2+m^2+n^2)\frac{dz}{dt}=0$$

or $\quad \dfrac{d^2p}{dt^2}=-k^2 p$, where $k^2=l^2+m^2+n^2$ and $p=\dfrac{dz}{dt}$

$$\therefore \quad \frac{dz}{dt}=p=A \sin kt+B \cos kt$$

or $\quad z=-(A/k) \cos kt+(B/k) \sin kt$
$\quad\quad =c_1 \cos kt+c_2 \sin kt.$

Similarly we get

$$x=a_1 \cos kt+a_2 \sin kt \text{ and } y=b_1 \cos kt+b_2 \sin kt.$$

The constants a_1, b_1, c_1, etc. are not independent. Substituting these values of x, y, z in the given equations, we get

$$(a_2k-nb_1+mc_1) \cos kt+(mc_2-a_1 k-nb_3) \sin kt+(mc_3-nb_3)=0.$$
$$(na_1+b_2 k-lc_1) \cos kt+(na_2-b_1 k-lc_2) \sin kt+(na_3-lc_3)=0.$$

and

$$(-ma_1+lb_1+c_2k) \cos kt+(-ma_2+lb_2-c_1k) \sin kt+(lb_3-mn_3)=0.$$

Equating the coefficients of cos kt, sin kt to zero, we get

$$k=(mc_1-nb_1)/a_2=(na_1-lc_1)/b_2=(lb_1-ma_1)/c_2 \quad (1)$$

$$=(mc_2-nb_2)/a_1=(na_2-lc_2)/b_1=(lb_2-ma_2)/c_1=\frac{0}{a_1l+b_1m+c_1n}$$

$$\therefore \quad a_1 l+b_1 m+c_1 n=0. \quad (2)$$

Also $\quad lb_3-ma_3=0=na_3-lc_3=mc_3-nb_3$ giving

$$a_3/l=b_3/m=c_3/n. \quad (3)$$

(1), (2), (3) give the relations between the constants a_1, b_1, c_1 etc.

Aliter. Multiplying (1), (2) and (3) by x, y, z respectively and adding we get

$$x \frac{dx}{dt}+y \frac{dy}{dt}+z \frac{dz}{dt}=0 \quad \therefore \quad x^2+y^2+z^2=a^2. \quad (1)$$

Similarly multiplying by l, m, n respectively, adding and integrating we get

$$lx+my+nz=\text{const}=b \sqrt{(l^2+m^2+n^2)} \quad \text{(say)}. \quad (2)$$

We shall now express $\dfrac{dx}{dt}$ in terms of functions of x. To do so we have

$$(ny-mz)^2=n^2y^2+m^2z^2-2mn\, yz=(m^2+n^2)(y^2+z^2)-(my+nz)^2$$
$$=(m^2+n^2)(a^2-x^2)-\{ b\sqrt{(l^2+m^2+n^2)}-lx \}^2$$
$$=(m^2+n^2)(a^2-b^2)-[(l^2+m^2+n^2$$
$$-2lbx\sqrt{l^2+m^2+n^2}+b^2 l^2]$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=(m^2+n^2)(a^2-b^2)-(kx-bl)^2,$$

Differentiating (1) with respect to t we get

$$t\frac{d^2x}{dt^3}+3\frac{dx}{dt}-2\frac{dy}{dt}=1. \qquad \ldots\ldots(3)$$

Eliminating y and $\frac{dy}{dt}$ in (1), (2) and (3) we easily get

$$t\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12\frac{x}{t}=2t+6$$

or $$t^2\frac{d^2x}{dt^2}+8t\frac{dx}{dt}+12x=2t^2+6t.$$

This is a homogeneous linear equation. Putting

$t=e^z$, $\frac{d}{dz}\equiv D$, we get

$$[D(D-1)+8D+12]\,x=2e^{2z}+6e^z$$

or $$(D^2+7D+12)\,x=2e^{2z}+6e^z$$

C. F. is $Ae^{-4z}+Be^{-3z}$ i.e. $At^{-4}+Bt^{-3}$.

For *P. I.*, $x=\dfrac{1}{(D+4)(D+3)}(2e^{2z}+6e^z)$ where $D\equiv\dfrac{d}{dz}$, $t=e^z$

$$=e^{2z}/15+3e^z/10=t^2/15+3t/10$$

$$\therefore\quad x=At^{-4}+Bt^{-3}+t^2/15+3t/10$$

Substituting these values in (1), we get

$$y=-At^{-4}-\tfrac{1}{2}Bt^{-3}+2t^2/15-t/20.$$

Ex. 9. *Solve :* $$\frac{dx}{dt}-ny+mz=0, \qquad \ldots\ldots(1)$$

$$nx+\frac{dy}{dt}-lz=0, \qquad \ldots\ldots(2)$$

$$-mx+ly+\frac{dz}{dt}=0. \qquad \ldots\ldots(3)$$

We have

$Dx-ny+mz=0$, $nx+Dy-lz=0$ and $-mx+ly+Dz=0$.

Solving the first two we get

$$x=\frac{nl-mD}{D^2+n^2}\cdot z \text{ and } y=\frac{mn+lD}{D^2+n^2}\cdot z$$

Substituting these in the third, we get

$$\frac{d^3z}{dt^3}+(l^2+m^2+n^2)\frac{dz}{dt}=0$$

12. $2\frac{d^2y}{dx^2}-\frac{dz}{dx}-4y=2x,\ 2\frac{dy}{dx}+4\frac{dz}{dx}-3z=0.$

(*Jaipur* 1952, 1955)

13. $t\frac{dx}{dt}+y=0,\ t\frac{dy}{dt}+x=0.$ (*Lucknow* 1954)

14. $\frac{d^2x}{dt^2}+4x+y=t\,e^{3t},\ \frac{d^2y}{dt^2}+y-2x=\cos^2 t.$

15. $t\frac{d^2x}{dt^2}+2\frac{dx}{dt}+tx=0,\ \frac{dy}{dt}+\frac{2}{t}y=\frac{dx}{dt}.$ (*Delhi* 1959)

16. $\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}$ where

$$X=ax+by+cz+d,\ Y=a'x+b'y+c'z+d'$$
$$Z=a''x+b''y+c''z+d''$$ (*Jaipur* 1957)

17. $lt\frac{dx}{dt}=mn(y-z),\ mt\frac{dy}{dt}=nl(z-x),\ nt\frac{dz}{dt}=lm(x-y).$

[**Hint.** Putting $lx=X,\ my=Y,\ nz=Z,\ t=e^T$, we get

$$\frac{dX}{dT}=nY-mZ,\ \frac{dY}{dT}=lZ-nX,\ \frac{dZ}{dT}=mX-lY.$$]

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9 3. Simultaneous equations of the form $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

Let us consider the equations of the type

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad ...(1)$$

where P, Q, R are the functions of x, y, z.

Method of Solution. The following suggestions will give us suitable method of solution :

1. By equating two of the three members of (1) we may be able to get an equation in only two variables. Sometimes such an equation is obtained after cancellation of some factor from the equation. The solution of this gives us one of the relations in the general solution of (1). This method may be repeated to give another relation with the help of two other members. The two relations, so found, give the complete solution of (1). [See solved example 10].

2. If one of the relations is known, or say found by the method given in (1) above, we may be able to use this in expressing one variable in terms of the others. This may help us in getting an equation in two variables and thus enable us to get another relation. [See solved example 11].

3. We may be able to find multipliers l, m, n and L, M, N such that one of the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l dx+m dy+n dz}{lP+mQ+nR}=\frac{L dx+M dy+N dz}{LP+MQ+NR}$$

can be easily integrated. (The numerators are differentials of the denominators or else the denominators are zero and numerators are exact differentials. Sometimes only one set of multipliers may serve our purpose). [See solved examples 12 and 13].

Any combination of the methods given above will help us in getting two independent relations between the variables, each containing an arbitrary constant. Hence we get the general solution of (1).

9·4. Simultaneous Equations in a different form. If the equations are given in the form

$$P_1\,dx+Q_1\,dy+R_1\,dz=0,$$
$$P_2\,dx+Q_2\,dy+R_2\,dz=0,$$

where the coefficients are functions of x, y, z they can be solved, by taking one of the variables, say x, as the independent variable for $\frac{dy}{dx}$ and $\frac{dz}{dx}$. Solving these we get

$$\frac{dy}{dx}=\frac{R_1\,P_2-P_1\,R_2}{Q_1\,R_2-Q_2\,R_1},\ \frac{dz}{dx}=\frac{P_1\,Q_2-P_2\,Q_1}{Q_1\,R_2-Q_2\,R_1}$$

whence $$\frac{dx}{Q_1R_2-Q_2R_1}=\frac{dy}{R_1P_2-R_2P_1}=\frac{dz}{P_1Q_2-P_2Q_1},$$

which is of the form

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R},$$

already discussed.

Ex. 10. *Solve* : $\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$.

Taking first two members

$$x\,dx=y\,dy$$

and its integral is $x^2-y^2=c_1$ ......(1)

Taking last two members

$$y\,dy = z\,dz$$

and its integral is $\quad y^2 - z^2 = c^2 \qquad \ldots\ldots(2)$

(1) and (2) constitute the complete integral.

Ex. 11. *Solve* : $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{z}$.

Taking first two members of the given equation we get on integration $\quad x^2 - y^2 = c_1. \qquad \ldots\ldots(1)$

Taking last two members eliminating x with the help of (1) the equation involving dy and dz becomes

$$\frac{dy}{\sqrt{(y^2+c_1)}} = \frac{dz}{z},$$

and the integral of this is

$$\log\,[y+\sqrt{(y^2+c_1)}] = \log z + c$$

which on simplification becomes

$$y + \sqrt{(y^2+c_1)} = c_2\,z. \qquad \ldots\ldots(2)$$

The integrals (1) and (2) from the complete integral of the set of equations.

Ex. 12. *Solve* : $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$ (*Raj.* 1958)

By using x, y, z as the multipliers each of these fractions is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

whence $\quad x\,dx + y\,dy + z\,dz = 0.$

Integrating $\quad x^2 + y^2 + z^2 = c_1. \qquad \ldots\ldots(1)$

Again using l, m, n as the multipliers each of these fractions is equal to

$$\frac{l\,dx + m\,dy + n\,dz}{0}$$

whence $\quad l\,dx + m\,dy + n\,dz = 0.$

Integrating $\quad lx + my + nx = c_2. \qquad \ldots\ldots(2)$

Equations (1) and (2) constitute the complete solutions of the given equations.

Ex. 13. *Solve* :

$$\frac{dx}{z\,(x+y)} = \frac{dy}{z\,(x-y)} = \frac{dz}{x^2+y^2}. \qquad (\textit{Agra } 1937)$$

Using $x, -y, -z$ as multipliers each fraction

$$=\frac{xdy-ydy-zdz}{0}.$$

Therefore, $xdx-ydy-zdz=0$.

Integrating, $x^2-y^2-z^2=c_1$. (1)

Using y, x, $-z$ as multipliers, each fraction

$$=\frac{ydx+xdy-zdz}{0}.$$

Therefore $ydx+xdy-zdz=0$.

Integrating $2xy-z^2=c_2$. (2)

(1) and (2) constitute the complete integral.

Ex. 14. *Solve* : $\dfrac{adx}{(b-c)\,yz}=\dfrac{bdy}{(c-a)\,zx}=\dfrac{cdz}{(a-b)\,xy}$

(*Raj.* 1951, *Nagpur* 1961)

Using x, y, z as multipliers each fraction

$$=\frac{ax\,dx+by\,dy+cz\,dz}{0}.$$

Therefore $ax\,dx+by\,dy+cz\,dz=0$.

Integrating $ax^2+by^2+cz^2=c_1$. (1)

Similarly using ax, by, cz as multipliers

$$a^2x\,dx+b^2y\,dy+c^2z\,dz=0.$$

Integrating $a^2x^2+b^2y^2+c^2z^2=c_2$. (2)

(1) and (2) form the complete integral.

Ex. 15. *Solve* : $\dfrac{dx}{z}=\dfrac{dy}{-z}=\dfrac{dz}{z^2+(y+x)^2}$.

Taking first two equations the integral is

$$x+y=c. \quad (1)$$

Using this relation in last two, we get

$$dy=\frac{-zdz}{z^2+c^2}$$

giving $y=-\frac{1}{2}\log(z^2+c^2)+c_2$

whence $z^2+c^2=A\,e^{-2y}$ i.e. $z^2+(y+x)^2=A\,e^{-2y}$ (2)

(1) and (2) give the complete integral.

Ex. 16. *Solve* : $\dfrac{dx}{xy}=\dfrac{dy}{y^2}=\dfrac{dz}{zxy-2x^2}$ (*Agra* 1959)

The first two equations give $x=ay$ ......(1)

Taking first and third equations

$$\frac{dx}{y}=\frac{dz}{zy-2x}.$$

With the help of (1) it reduces to

$$\frac{dx}{y}=\frac{dz}{zy-2ay} \quad i.e. \quad dx=\frac{dz}{z-2a}.$$

Integrating, $x=\log(z-2a)+b$

$$x=\log(z-2x/y)+b \qquad \ldots\ldots(2)$$

Hence (1) and (2) give the complete solution.

Ex 17. *Solve* : $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$

Each of the fractions $=\dfrac{dx+dy+dz}{0}$,

whence $\quad x+y+z=a. \qquad \ldots\ldots(1)$

Using $1/x$, $1/y$, $1/z$ as multipliers, each of these fractions

$$=\frac{dx/x+dy/y+dz/z}{0},$$

whence $\quad \log x+\log y+\log z=\text{constant}.$

i.e. $\quad xyz=b. \qquad \ldots\ldots(2)$

(1) and (2) constitute the complete integrals.

Ex. 18. *Solve* : $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{y(z^2-x^2)}=\dfrac{z}{z(x^2-y^2)}.$

Using x, y, z as multipliers we get

$$xdx+ydy+zdz=0$$

whence $\quad x^2+y^2+z^2=a. \qquad \ldots\ldots(1)$

Using $1/x$, $1/y$, $1/z$ as multipliers we get

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

whence $\quad xyz=b. \qquad \ldots\ldots(2)$

(1) and (2) constitute the complete integrals.

(v. imp) **Ex. 19.** *Solve* : $\dfrac{dx}{\cos(x+y)}=\dfrac{dy}{\sin(x+y)}=\dfrac{dz}{z}.$

Here it is obvious that by taking proper multipliers

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

or $\dfrac{[\cos(x+y)-\sin(x+y)]\,(dx+dy)]}{\cos(x+y)+\sin(x+y)}=dx-dy$

Integrating $\log[\cos(x+y)+\sin(x+y)]=x-y+\text{constant}.$

or $[\cos(x+y)+\sin(x+y)]\,e^{y-x}=a.$ ...(1)

Also $$\frac{dz}{z}=\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{(1/\sqrt{2})\,(dx+dy)}{\sin(x+y+\frac{1}{4}\pi)}$$

$$\therefore\quad \sqrt{2}\,\frac{dz}{z}=\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)(dx+dy)$$

$$\sqrt{2}\log z=\log\tan\left\{\tfrac{1}{2}(x+y)+\frac{\pi}{8}\right\}+\log b$$

$$\therefore\quad z^{\sqrt{2}}\cot\left\{\tfrac{1}{2}(x+y)+\frac{\pi}{8}\right\}=b. \qquad ...(2)$$

Ex. 20. *Solve* : $\dfrac{dx}{y^3x-2x^4}=\dfrac{dy}{2y^4-x^3y}=\dfrac{dz}{9z\,(x^3-y^3)}$.

(*Agra* 1941, *Rajasthan* 1955)

$$\frac{dx/x}{y^3-2x^3}=\frac{dy/y}{2y^3-x^3}=\frac{dz/3z}{3\,(x^3-y^3)}=\frac{dx/x+dy/y+dz/3z}{0}$$

$$\therefore\quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z}=0,$$

Integrating $x\,y\,z^{1/3}=a.$ ...(1)

Again we have

$$(2y^4-x^3y)\,dx+(2x^4-y^3x)\,dy=0$$

Dividing by x^3y^3,

$$\frac{2y}{x^3}\,dx-\frac{dx}{y^2}+\frac{2x}{y^3}\,dy-\frac{dy}{x^2}=0$$

$$\therefore\quad \left(\frac{dy}{x^2}-\frac{2y}{x^3}\,dx\right)+\left(\frac{dx}{y^2}-\frac{2x}{y^3}\right)dy=0$$

$$\therefore\quad \frac{y}{x^2}+\frac{x}{y^2}=b. \qquad ...(2)$$

(1) and (2) together give the solution.

9·5. Geometrical Interpretation of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ (A)

We know from solid Geometry that the direction cosines of the tangent to a curve are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ *i.e.* are in the ratio of

$$dx : dy : dz.$$

The equations (*A*) therefore express that the tangent to a curve at any point (x, y, z) has direction cosines proportional to $P : Q : R$.

[For any point (x_1, y_1, z_1), equations (*A*) determine definite values of $\frac{dz}{dx}$ and $\frac{dy}{dx}$ and thus these differential equations determine a particular direction at every point in space. Hence if a point (x_1, y_1, z_1) moves so that at any moment its coordinates and the direction ratios of its line of motion [dx, dy, dz] satisfy the given differential equations (*A*), then the point (x_1, y_1, z_1) must pass through each position in a particular direction. Consequently, a point, which starts at a given point and moves in the direction determined for this point by the differential equations (*A*), will describe a curve in space passing through the point (x_1, y_1, z_1). The coordinates of any point on this curve and direction of the curve at that point satisfy the given differential equations (*A*).

Next, take a point (x_2, y_2, z_2) which does not lie on the curve obtained above by the motion of (x_1, y_1, z_1); this point will describe another curve. Thus through every point in space, we will have a definite curve whose equation will satisfy the given differential equations (*A*).

The curves are the curves obtained by the intersection of the two surfaces given by the two independent integrals of (*A*).]

EXERCISE IX (B)

Solve :

1. $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$.

2. $\frac{x\,dx}{y^2 z}=\frac{dy}{xz}=\frac{dz}{y^2}$.

3. $\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$:

4. $\frac{dx}{y^2}=\frac{dy}{x^2}=\frac{dz}{x^2y^2z^2}$.

5. $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$:

6. $\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$.

7. $\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{n\,xy}$.

8. $\frac{dx}{x}=\frac{dy}{z}=-\frac{dz}{y}$.

9. $\frac{l\,dx}{mn\,(y-z)}=\frac{m\,dy}{nl\,(z-x)}=\frac{n\,dz}{lm\,(x-y)}$.

10. $\dfrac{dx}{1}=\dfrac{dy}{-2}=\dfrac{dz}{3x^2 \sin (y+2x)}$.

11. $\dfrac{x\,dx}{z^2-2yz-y^2}=\dfrac{dy}{y+z}=\dfrac{dz}{y-z}$.

12. $\dfrac{dx}{1}=\dfrac{dy}{3}=\dfrac{dz}{5z+\tan (y-3x)}$. (*Bombay* 1961)

13. $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}$.

14. $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{-y(z^2+x^2)}=\dfrac{dz}{z(x^2+y^2)}$.

15. $\dfrac{dx}{xz(z^2+xy)}=\dfrac{dy}{-yz(z^2+xy)}=\dfrac{dz}{x^4}$.

16. $\dfrac{-dx}{x(x+y)}=\dfrac{dy}{y(x+y)}=\dfrac{dz}{(x-y)(2x+2y+z)}$.

17. $\dfrac{dx}{y^2+yz+z^2}=\dfrac{dy}{z^2+zx+x^2}=\dfrac{dz}{x^2+xy+y^2}$. (*Nagpur* 1958)

18. $\dfrac{dx}{y}=\dfrac{dy}{x}=\dfrac{dz}{xyz^2(x^2-y^2)}$. (*Bombay* 1961)

CHAPTER X

TOTAL DIFFERENTIAL EQUATIONS

10·1. If we are given a relation of the form

$$\phi\ (x, y, z)=c \qquad \text{...(1)}$$

where c is an arbitrary constant, we can easily get

$$\frac{\partial \phi}{\partial x} dx+\frac{\partial \phi}{\partial y} dy+\frac{\partial \phi}{\partial z} dz=0. \qquad \text{...(2)}$$

If $\dfrac{\partial \phi}{\partial x}$, $\dfrac{\partial \phi}{\partial y}$, $\dfrac{\partial \phi}{\partial z}$ have a common factor, the relation (2) can be simplified by cancelling the factor. In any case the derived equation (2) can be expressed in the form

$$P\ dx+Q\ dy+R\ dz=0, \qquad \text{...(4)}$$

where P, Q R are functions of x, y, z.

Thus if we are given relation (1), we can always get a relation of the form (4).

We shall now derive the *converse* of it *i.e.* if we are given *any* relation of the form (4), how to obtain relation (1). This is not possible if P, Q, and R are taken arbitrary. In the next article we shall find under what circumstances such a differential equation will lead to an integral of the form (1).

A relation of the type $Pdx+Qdy+Rdz=0$ *where* P, Q, R *are functions of* x, y, z *is called a* **total differential equation.**

10·2. Condition of Integrability. Let us consider the single differential equation

$$P\ dx+Q\ dy+R\ dz=0, \qquad \text{...(1)}$$

where P, Q, R are functions of x, y, z. Let it, if it is integrable, have an integral

$$\phi=\text{constant}=a, \text{ say} \qquad \text{...(2)}$$

so that $d\phi$, the total differential of ϕ is equal to

$$P\ dx+Q\ dy+R\ dz$$

or equal to it multiplied by a factor. But

$$d\phi\equiv\frac{\partial \phi}{\partial x}\ dx+\ \frac{\partial \phi}{\partial y}\ dy+\frac{\partial \phi}{\partial z}\ dz=0. \qquad \text{...(3)}$$

Since (2) gives the integral of (1) we must have

$$\frac{\partial \phi}{dx}=\mu P,\ \frac{d\phi}{\partial y}=\mu Q,\ \frac{\partial \phi}{dz}=\mu R$$

where μ is some function the value of which is unknown and which is common to all.

From the first two of these equations we have

$$\frac{\partial}{\partial y}(\mu P)=\frac{\partial^2 \phi}{\partial x\,\partial y}=\frac{\partial}{\partial x}(\mu Q),$$

or $$\mu\frac{\partial P}{dy}+P\frac{\partial \mu}{\partial y}=\mu\frac{\partial Q}{dx}+Q\frac{\partial \mu}{dx},$$

that is $$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{dy}. \quad ...(4)$$

Similarly, $$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z}, \quad ...(5)$$

and $$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x}. \quad ...(6)$$

Multiplying (4) by R, (5) by P, (6) by Q and adding, we get

$$P\left(\frac{\partial Q}{dz}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

This is the relation which must exist between P, Q, R if (1) posseses an integral of the form considered.

The condition is therefore necessary ; it is also sufficient *i.e.* when this relation is satisfied, the given equation has a primitive of the form $\phi=a$. We proceed to prove this proposition.

If this relation is satisfied for P, Q, R of the equation

$$P\,dx+Q\,dy+R\,dz=0, \quad ...(1)$$

then it can be easily verified that the same relation will hold for the coefficients of

$$\mu P\,dx+\mu Q\,dy+\mu R\,dz=0 \quad ...(2)$$

where μ is any function of x, y, z. Let

$$P\,dx+Q\,dy$$

be an exact differential equation. In case it is not, an integrating factor can be found to make it exact. Hence there is no loss of generality in regarding $P\,dx+Q\,dy$ as an exact differential.

Let $\int (P\,dx+Q\,dy)=V$

so that $$\frac{\partial V}{\partial x}=P \text{ and } \frac{\partial V}{\partial y}=Q.$$

Further condition of being exact is

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.$$

Hence $\quad \dfrac{\partial P}{\partial z}=\dfrac{\partial^2 V}{\partial z\,\partial x}, \quad \dfrac{\partial Q}{\partial z}=\dfrac{\partial^2 V}{\partial z\,\partial y}$

Substituting these values in the given relation

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

it becomes

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z\,\partial x}\right)=0$$

and it may be written in the form

$$\frac{\partial V}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}-R\right)-\frac{\partial V}{\partial y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}-R\right)=0$$

$$\text{or}\quad \begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z}-R\right) \\[2ex] \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z}-R\right) \end{vmatrix}=0$$

This equation shows that a relation independent of x and y exists between V and $\dfrac{\partial V}{\partial z}-R$. Therefore $\dfrac{\partial V}{\partial z}-R$ can be expressed as a function of z and V alone

Let $\quad \dfrac{\partial V}{dz}-R=f(z, V)$. $\quad$ ...(4)

Now $\quad P\,dx+Q\,dy+R\,dz$

$$=\frac{\partial V}{\partial x}\,dx+\frac{\partial V}{\partial y}\,dy+\frac{\partial V}{\partial z}\,dz+\left(R-\frac{\partial V}{\partial z}\right)dz$$

$$=dV-f(z, V)\,dz.$$

This being an equation in two variables, we may have its integral of the form

$$F(z, V)=0$$

Hence the condition is sufficient.

10·3. Method to obtain the primitive of $P\,dx+Q\,dy+R\,dz=0$.

In case the condition of integrability is satisfied, we integrate the equation

$$P\,dx+Q\,dy=0^*$$

as if z were constant *i.e.* $dz=0$. Let this integral be

$$u=\text{constant}=\phi \qquad \ldots\ldots(1)$$

This constant in the integral is constant only with respect to x and y and can therefore be taken as a function of z.

On differentiating the integral (1) with respect to x, y, z and comparing the result with

$$P\,dx+Q\,dy+R\,dz=0$$

we shall be able to get a relation independent of dx and dy. If the cofficients of $d\phi$ or dz involve functions of x and y it will be possible to eliminate them with the help of (1). Then we shall determine ϕ by integration.

10·4. Solution by inspection.

When the condition of integrability is satisfied it will be possible, in many cases, to get the solution by inspection. In such cases it will be unnecessary to make use of the general method.

Ex. 1. *Solve* : $(y+z)\,dx+dy+dz=0.$ (*Agra* 1954)

We have $dx+\dfrac{dy+dz}{y+z}=0$

$$x+\log(y+z)=\log c$$
$$y+z=c\,e^{-x}.$$

Ex. 2. *Solve* : $yz\log z\,dx-zx\log z\,dy+xy\,dz=0.$ (*Agra* 1939)

Arranging the terms we have

$$\frac{ydx-xdy}{xy}+\frac{dz/z}{\log z}=0$$

or $$\frac{dx}{x}-\frac{dy}{y}+\frac{dz/z}{\log z}=0$$

or $$\log c+\log x-\log y+\log(\log z)=0$$

$$\therefore\ y=cx\log z.$$

Ex. 3. *Solve* : $(2x^2+2xy+2xz^2+1)\,dx+dy+2zdz=0.$ (1)
(*Nagpur* 1961, *Raj*, 1957, *Agra* 1947)

Keeping x constant

$$y+z^2=\phi \qquad (2)$$
$$dy+2zdz=d\phi \qquad (3)$$

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*We may as well proceed by assuming

$$Qdy+Rdz=0 \quad \text{or} \quad Pdx+Rdz=0$$

and making the necessary changes in the above statement.

Comparing (1) and (3)

$$[\,(2x^2+1)+2x\,(y+z^2)\,]\,dx=-d\phi.$$

Thus we get a relation independent of dy and dz.
Using (2) we get

$$[\,(2x^2+1)+2x\phi\,]\,dx=-d\phi$$

Thus terms involving functions of y and z are removed.

$$\therefore\quad \frac{d\phi}{dx}+2x\phi=-(2x^2+1)$$

$$\phi\, e^{x^2}=-\int(2x^2+1)\,e^{x^2}\,dx$$

Now $\quad \int 2x^2\,e^{x^2}dx=x\,e^{x^2}-\int e^{x^2}\,dx$

$$\int(2x^2+1)\,e^{x^2}dx=x\,e^{x^2}$$

The integral is

$$\phi\, e^{x^2}=-x\,e^{x^2}+c$$

or $\quad e^{x^2}\,(y+z^2+x)=c.$

Ex. 4. *Solve* : $(yz+xyz)\,dx+(zx+xyz)\,dy+(xy+xyz)dz=0$

(*Raj*, 1953, 1955, *Agra* 1958)

Dividing the whole equation by xyz it becomes

$$(1/x+1)\,dx+(1/y+1)\,dy+(1/z+1)\,dz=0.$$

Integrating, we have

$$\log x+x+\log y+y+\log z+z=\text{constant}$$
$$\log xyz+x+y+z=c.$$

Ex. 5. *Solve* : $zdz+(x-a)\,dx=[\,h^2-z^2-(x-a)^2\,]^{1/2}\,dy.$

(*Agra* 1947)

We have $\dfrac{z\,dz+(x-a)\,dx}{[\,h^2-z^2-(x-a)^2]^{1/2}}=dy$

Integrating $[\,h^2-z^2-(x-a)^2\,]^{1/2}=c-y$

$$\therefore\quad h^2-z^2-(x-a)^2=(c-y)^2.$$

Ex. 6. *Solve* : $y^2\,z(x\cos x-\sin x)dx+x^2\,z(y\cos y-\sin y)dy$
$+xy\,(y\sin x+x\sin y+xy\cos z)\,dz=0.$

It can be easily shown that the condition of integrability is satisfied. Putting $z=$constant and therefore $dz=0$, the equation becomes

$$\frac{(x\cos x-\sin x)\,dx}{x^2\,z}+\frac{y\cos y-\sin y}{y^2\,z}\,dy=0$$

Integrating $\dfrac{\sin x}{xz}+\dfrac{\sin y}{yz}=\phi$

where ϕ is some function of z.

Differentiating (1) and comparing with the given equation we get

$$x\,y\,z\,d\phi=-(y\sin x+x\sin y+x\,y\cos z)\,dz$$

or $\quad z\,d\phi=-\left(\dfrac{\sin x}{x}+\dfrac{\sin y}{y}+\cos z\right)dz=-(\phi+\cos z)\,dz$

or $\quad \dfrac{d\phi}{dz}+\dfrac{1}{z}\,\phi=-\dfrac{\cos z}{z}$

Integrating $\phi z=-\sin z+c$ or $\phi=\dfrac{-\sin z+c}{z}$.

$\therefore$ The integral is $z\left(\dfrac{\sin x}{x}+\dfrac{\sin y}{y}\right)+\sin z=c.$

Ex. 7. *Solve :* $(ydx+xdy)(a-z)+xy\,dz=0$ (*Agra* 1936)

We have $\dfrac{y\,dx+x\,dy}{xy}+\dfrac{dz}{a-z}=0.$

Integrating, $\log(x\,y)-\log(a-z)=\log c$

$\therefore\quad xy=c\,(a-z).$

Ex. 8. *Solve :*

$$\frac{yz}{x^2+y^2}dx-\frac{xz}{x^2+y^2}\,dy-\tan^{-1}\frac{y}{x}\,dz=0,$$

We have $\dfrac{y\,dx-x\,dy}{(x^2+y^2)\tan^{-1}y/x}=\dfrac{dz}{z}$ ...(1)

Let $v=\tan^{-1}\dfrac{y}{x}$, $\therefore\quad dv=\dfrac{(x\,dy-y\,dx)}{1+y^2/x^2}$.

Then (1) becomes $-\dfrac{dv}{v}=\dfrac{dz}{z}$

Integrating $\quad -\log v=\log cz$

or $\quad 1/v=cz$ *i e.* $y/x=\tan(1/cz)$.

EXERCISE X (A)

Solve the following differential equations :

1. $dx+dy+dz=0.$
2. $xdx+y\,dy+z\,dz=0.$
3. $yzdx+zxdy+xydz=0.$
4. $zydx=zxdy+y^2\,dz.$

5. $xdx+zdy+(y+2z)\,dz=0.$ (*Raj.* 1959)
6. $(y+z)\,dx+(z+x)\,dy+(x+y)\,dz=0.$ (*Punjab* 1955)
7. $yz\,dx+2zx\,dy-3xy\,dz=0.$
8. $(a+z)\,ydx+(a+z)\,xdy-xy\,dz=0.$
9. $a\,y^2\,z^2\,dx+b\,z^2\,x^2\,dy+c\,x^2y^2\,dz=0.$
10. $dx+dy+(x+y+z+1)\,dz=0.$
11. $(yz+2x)\,dx+(xz+2y)\,dy+(xy+2z)\,dz=0.$
12. $(y^2+z^2-x^2)\,dx-2xy\,dy-2xz\,dz=0.$
13. $2yz\,dx+zx\,dy-xy\,(1+z)\,dz=0.$
14. $(y+b)\,(z+c)\,dx+(x+a)\,(z+c)\,dy+(x+a)\,(y+b)\,dz=0.$
15. $\dfrac{x\,dx+y\,dy+z\,dz}{\sqrt{(x^2+y^2+z^2)}}+\dfrac{zdx-xdz}{x^2+z^2}+3a\,x^2\,dx+2by\,dy+cdz=0.$
16. $(y+a)^2\,dx+z\,dy-(y+a)\,dz=0.$
17. $(ay-bz)\,dx+(cz-ax)\,dy+(bx-cy)\,dz=0.$
18. $(mz-ny)\,dx+(nx-lz)\,dy+(ly-mx)\,dz=0.$
19. $2\,(y+z)\,dx+(x+3y+2z)\,dy+(x+y)\,dz=0.$
20. $(y-z)\,dx+2\,(x+3y-z)\,dy-2\,(x+2y)\,dz=0.$
21. $(x-3y-z)\,dx+(2y-3x)\,dy+(z-x)\,dz=0.$
22. $(2y^2+4a\,z^2\,x^2)\,xdx+\{\,3y+2x^2+(y^2+z^2)^{-1/2}\,\}\,ydy$
$+[\,4z^3+2ax^4+(y^2+z^2)^{-1/2}\,]\,zdz=0.$
23. $(2x^2y+2x\,y^2+2xyz+1)\,dx+(x^3+x^2\,y+x^2\,z+2xyz$
$+2y^2z+2yz^2+1)\,dy+(xy^2+y^3+y^2\,z+1)\,dz=0.$
24. Find $f(y)$ if $f(y)\,dx-z\,x\,dy-xy\log y\,dz=0$ is integrable. Find the corresponding integral.

Solve :

25. $(z+z^2)\cos x\,\dfrac{dx}{dt}-(z+z^2)\,\dfrac{dy}{dt}+(1-z^2)\,(y-\sin x)\,\dfrac{dz}{dt}=0.$
26. $(x^2+z^2)\,(x\,dx+y\,dy+z\,dz)+\sqrt{x^2+y^2+z^2}\,(zdx-xdz)=0.$

10·5. Homogeneous Equations. In the case of a homogeneous equation between x, y, z, one variable can be separated from the other two by substitution

$$x=zu,\; y=zv$$

and therefore $dx=z\,du+u\,dz,\; dy=z\,dv+v\,dz.$

Then the equation will reduce to

$$z\,\phi(u,v)\,dx+z\,\psi\,(u,v)\,dv+[\,\theta(u,v)+u\,\phi(u,v)+v\,\psi\,(u,v)\,]\,dz=0.$$

Case I. If the coefficient of dz is zero, we have an equation between the two variables u and v and it can, therefore, be easily integrated.

Case II. If the coefficient of dz is not zero, the equation will take the form

$$\frac{\phi\,(u,v)\,du+\psi\,(u,v)\,dv}{\theta\,(u,v)+u\,\phi\,(u,v)+v\,\psi\,(u,v)}+\frac{dz}{z}=0.$$

Here if the given equation is integrable, the first term will be an exact differential.

Ex. 9. *Solve :* $(y^2+yz)\,dx+(xz+z^2)\,dy+(y^2-xy)\,dz=0.$
(*I.A.S.* 1961, *Agra*,'51,'60; *Jaipur* 1949,'51,'53,'56; *Luck*,1948,'54,'55)

Here the condition of integrability is satisfied. The equation is homogeneous, hence substituting

$$x=uz,\ y=vz$$

and therefore $dx=zdu+udz$ and $dy=zdv+vdz$

the given equation is reduced to

$$(v^2+v)\,z\,du+z\,(u+1)\,dv+(1+u)\,(v^2+v)\,dz=0$$

or $$\frac{du\,(v+v^2)+dv\,(u+1)}{(1+u)\,(v^2+v)}+\frac{dz}{z}=0$$

or $$\frac{du\,(v+v^2)+dv\,(2v+1)\,(1+u)}{(1+u)\,(v^2+v)}-\frac{2v\,(u+1)\,dv}{(1+u)\,(v^2+v)}+\frac{dz}{z}=0$$

Integrating

$$\log\{(1+u)\,(v^2+v)\}-\log(v+1)^2+\log z+\log c'=0$$

or $$\frac{(1+u)\,(v^2+v)\,z}{(v+1)^2}=c \quad \text{or} \quad \frac{(x+z)\,y}{y+z}=c$$

or $y\,(x+z)=c\,(y+z)$.

10·6. The method of finding an integrating factor for Homogeneous Equations.

Let $$P\,dx+Q\,dy+R\,dz=0 \tag{1}$$
be an integrable equation in which P, Q, R are homogeneous functions of degree n in x, y, z.

Substitute $x=uz$ and $y=vz$ and let

$$P=z^n f(u,v),\ Q=z^n F(u,v),\ R=z^n \phi(u,v).$$

Then $dx=udz+zdu$, $dy=vdz+zdv$.

Putting these values in (1) it becomes

$$z^n\,[\,f(u,v)\,(udz+zdv)+F(u,v)\,(vdz+zdv)+\phi(u,v)\,dz\,]=0$$

i.e. $$z^n\,[\,z\,\{f(u,v)\,du+F(u,v)\,dv\}+\{u\,f(u,v)+v\,F(u,v)+\phi(u,v)\}\,dz\,]=0.$$

Dividing by $z^{n+1}\,[\,u\,f(u,v)+v\,F(u,v)+\phi(u,v)\,]$, if this expression is not zero we get

$$\frac{f(u,v)\,du+F(u,v)\,dv}{u\,f(u,v)+v\,F(u,v)+\phi(u,v)}+\frac{dz}{z}=0 \tag{2}$$

Since equation (1) is integrable so is equation (2) either immediately or after multiplication by an integrating factor. In the first term of equation (2) the variables are u and v while in the second term the variable is only z. Thus the variable z is separated

from the other two. In case we multiply (2) by any factor, other than a constant, this separation would be destroyed. So there is no integrating factor and therefore equation (2) must be exact in itself. But equation (2) was obtained from equation (1) by dividing it by the factor

$$z^{n+1}\,[\,u\,f(u, v)+v\,F\,(u, v)+\phi\ (u, v)\,]$$

besides the change of variable; and this factor is equal to $Px+Qy+Rz$. Hence $1/(Px+Qy+Rz)$ is an integrating factor of the integrable homogeneous equation

$$Pdx+Qdy+Rdz=0,$$

except when $\quad Px+Qy+Rz=0.$

Note :—*Similarly it can be shown that* $1/(P_1\,x_1+P_2\,x_2+\dots\dots\dots+P_n\,x_n)$ *is an integrating factor for*

$$P_1\,dx_1+P_2\,dx_2+\dots\dots\dots\dots+P_n\,dx_n=0,$$

except when $P_1\,x_1+P_2\,x_2+\dots\dots\dots\dots+P_n\,x_n=0.$

Ex. 10. *Solve* : $(yz+z^2)\,dx-xz\,dy+xy\,dz=0.$

Here $Px+Qy+Rz=xyz+xz^2-xyz+xyz$

$$=xz\,(y+z)=D, \text{ say}$$

$\therefore\quad d\,(D)=xz\,(dy+dz)+(y+z)\,(xdz+zdx)$

Dividing the given equation by D, we get

$$\frac{z\,(y+z)\,dx-xz\,dy+xy\,dz}{D}=0$$

or $\quad \dfrac{d\,(D)}{D}-\dfrac{2xz\,(dy+dz)}{xz\,(y+z)}=0,\quad$ or $\quad \dfrac{d\,(D)}{D}-\dfrac{2\,(dy+dz)}{y+z}=0$

Integrating $\quad D=c\,(y+z)^2$

or $\quad xz=c\,(y+z).$

Ex. 11. *Solve* :

$$(x^2y-y^3-y^2z)\,dx+(xy^2-x^2z-x^3)\,dy+(xy^2+x^2y)\,dz=0.$$

(*Agra* 1945, 1957, *Raj.* 1952)

The condition of integrability is satisfied. The equation is homogeneous [Here $Px+Qy+Rz=0$] hence substituting $x=uz$, $y=vz$, so that

$$dx=z\,du+u\,dz,\ dy=v\,dz+z\,dv,$$

the equation is reduced to

$$v\,(u^2-v^2-v)\,du+u\,(v^2-u-u^2)\,dv=0$$

or $\quad (u^2-v^2)\,(v\,du-u\,dv)-v^2\,du-u^2\,dv=0$

or $\quad (1/v^2-1/u^2)\,(v\,du-u\,dv)-(1/u^2)\,du-(1/v^2)\,dv=0$

Integrating we get

$$u/v+v/u+1/u+1/v=c$$

or $\quad x/y+y/x+z/x+z/y=c$

or $\quad x^2+y^2+yz+xz=cxy.$

Aliter. Divide by $x^2 y^2$

$$\frac{dx}{y}-\frac{y}{x^2}\,dy-\frac{z}{x^2}\,dx+\frac{dy}{x}-\frac{z\,dy}{x}-\frac{xdy}{y^2}+\frac{dz}{x}+\frac{dz}{y}=0$$

$$\therefore\quad \frac{ydx-xdy}{y^2}+\frac{xdy-ydx}{x^2}+\frac{xdz-zdx}{x^2}+\frac{ydz-zdy}{y^2}=0$$

$$x/y+y/x+z/x+z/y=c.$$

Ex. 12. *Solve* :

$$(y^2+yz+z^2)\,dx+(x^2+xz+z^2)\,dy+(x^2+xy+y^2)\,dz=0,$$

(*Punjab* '56, *I. A. S* '58, *Jaipur* '50 '54, 58, *Mysore* '49, *Agra* 1933, 1952, 1961 *Nagpur* '57 '58)

Here $P=y^2+yz+z^2$, $Q=x^2+xz+z^2$, $R=x^2+xy+y^2$
Substituting these values in the condition of integrability *i.e.*

$$P\left(\frac{\partial Q}{\partial y}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$=(y^2+yz+z^2)\,(x+2z-x-2y)+(x^2+xz+z^2)\,(2x+y-y-2z)$$
$$+(x^2+xy+y^2)\,(2y+z-2x-z)$$

$$=2\,(z^3-y^3+x^3-z^3+y^3-x^3)=0.$$

Hence the condition of integrability is satisfied. The equation is homogeneous, hence substituting $x=uz$, $y=vz$, and therefore $dx=u\,dz+z\,du$ and $dy=v\,dz+z\,dv$, the given equation is reduced to

$$\frac{(v^2+v+1)\,du+(u^2+u+1)\,dv}{(u+v+1)\,(uv+u+v)}+\frac{dz}{z}=0$$

If we write D for the denominator of the first term, the above expression becomes

$$\frac{d\,(D)-2\,(u+v+uv)\,(du+dv)}{D}+\frac{dz}{z}=0$$

or $$\frac{d\,(D)}{D}-\frac{2\,(du+dv)}{u+v+1}+\frac{dz}{z}=0$$

Integrating,

or $$\frac{u+v+uv}{u+v+1}\,z=c$$

or $$\frac{xz+yz+xy}{z^2}\times\frac{z}{x+y+z}\times z=c$$

or $$xy+yz+zx=c\,(x+y+z).$$

Aliter. We have

$$Px+Qy+Rz=(x+y+z)(xy+yz+zx)=D \quad \text{(say)}$$

$$\therefore \quad d(D)=(yz+zx+xy)(dx+dy+dz)$$
$$+(x+y+z)(ydz+zdy+zdx+xdz+xdy+ydx)$$

Dividing by D the given equation becomes

$$\frac{(y^2+yz+z^2)\,dx+\ldots\ldots\ldots\ldots}{D}=0$$

or $$\frac{dD}{D}-\frac{2(yz+zx+xy)(dx+dy+dz)}{(yz+zx+xy)(x+y+z)}=0$$

or $$\frac{dD}{D}-2\,\frac{dx+dy+dz}{x+y+z}=0$$

Integrating we get

$$D=c(x+y+z)^2$$
or $$xy+yz+zx=c(x+y+z).$$

EXERCISE X (B)

Solve :

1. $z^2\,dx+(z^2-2yz)\,dy+(2y^2-yz-xz)\,dz=0$. (*Agra* 1935, '41, '50)
2. $(2xz-yz)\,dx+(2yz-xz)\,dy-(x^2-xy+y^2)\,dz=0$. (*Agra* 1948, 1953, 1959)
3. $(2xy+z^2)\,dx+(x^2+2yz)\,dy+(y^2+2xz)\,dz=0$.
4. $(2xyz+y^2z+yz^2)\,dx+(x^2z+2xyz+xz^2)\,dy+(x^2y+xy^2+2xyz)\,dz=0$.
5. $yz^2(x^2-yz)\,dx+x^2z(y^2-xz)\,dy+xy^2(z^2-xy)\,dz=0$. (*Nagpur* 1953)
6. $(x^2-y^2-z^2+2xy+2xz)\,dx+(y^2-z^2-x^2+2yz+2yx)\,dy+(z^2-x^2-y^2+2zx+2zy)\,dz=0$.
7. $2(2y^2+yz-z^2)\,dx+x(4y+z)\,dy+x(y-2z)\,dz=0$.

10·7. Geometrical interpretation of P dx+Q dy+R dz=0.

This differential equation represents that two straight lines of which the direction cosines are proportional to dx, dy, dz and P, Q, R are perpendicular to each other. Direction cosines of the tangent at a point (x, y, z) on a curve are proportional to dx, dy, dz. Hence the above equation expresses that the tangent to a curve at the point (x, y, z) is perpendicular to a line of which *d. c.*'s are proportional to P, Q, R.

Let the solution of

$$Pdx+Qdy+Rdz=0 \qquad \ldots\ldots(1)$$

be obtained as

$$f(x, y, c)=0. \qquad \ldots\ldots(2)$$

Since (2) has one arbitrary constant, it represents a singly infinite system of surfaces. By a suitable choice of this constant, the surface given by (2) can be made to pass through any given point of space. If the point moves on the surface *in any direction*, then its coordinates and the direction ratios dx, dy, dz of its path at *any moment* must satisfy (1) since (2) is the integral of (1). Also for each given point (x_1, y_1, z_1) there can be an infinite set of values of dx, dy, dz depending on the path followed by the point in moving from the initial position (x_1, y_1, z_1), and all these sets will satisfy equation (1). However, while describing any path the point must remain on the surface given by the integral (2), which of course, passes through the point (x_1, y_1, z_1) Thus we have an *infinite number* of curves thus described by a point which all lie on the surface (2). [Compare it with the case of Art. 9·5 where there can be only one curve. Here the number of curves is infinite but all of them must lie on one surface].

10·8. The locus of $\mathbf{P\,dx+Q\,dy+R\,dz=0}$ is orthogonal to the locus of $\dfrac{\mathbf{dx}}{\mathbf{P}}=\dfrac{\mathbf{dy}}{\mathbf{Q}}=\dfrac{\mathbf{dz}}{\mathbf{R}}$.

It has been shown above that if a point on a curve is moving subject to the condition

$$P\ dx+Q\ dy+R\ dz=0 \tag{1}$$

that the direction of the point is at right angles to a line whose *d. c's*. are proportional to P, Q, R.

It has also been shown that a straight line whose direction cosines are proportional to dx, dy, dz is parallel to a line whose direction cosines are proportional to P, Q, R under the condition

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \tag{2}$$

Hence the curves traced out by points that are moving subject to the condition (1) are orthogonal to the curves traced out by points that are moving subject to the condition (2).

Case I. In case the equation (1) is integrable, a family of surfaces can be found, which are normal to the curves represented by (2) at the points where these curves cut the surface.

Case II. In case the equation (1) is not integrable, no family of surface can be found which is orthogonal to all lines that form the locus of equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

10·9. Equations containing more than three variables. Let us consider an equation of the form

$$Pdx+Qdy+Rdz+Tdt=0.$$

It must obviously be integrable when any one of the four variables is made constant. Thus taking x, y, z, t successively as constants, the four conditions of integrability are

$$T\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial t}-\frac{\partial T}{\partial z}\right)+R\left(\frac{\partial T}{\partial y}-\frac{\partial Q}{\partial t}\right)=0.$$

$$T\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial t}-\frac{\partial T}{\partial x}\right)+P\left(\frac{\partial T}{\partial z}-\frac{\partial R}{\partial t}\right)=0$$

$$T\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)+P\left(\frac{\partial Q}{\partial t}-\frac{\partial T}{\partial y}\right)+Q\left(\frac{\partial T}{\partial x}-\frac{\partial P}{\partial t}\right)=0$$

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

It should be noted that the fourth condition can be derived from first three, hence is not independent.

Hence if an equation involves more than three variables, the condition of integrability must hold for the coefficients of all the terms taken by threes. Of course all of them may not be independent.

In the case of n independent variables the number of independent equations of condition will be found to be

$$\tfrac{1}{2}(n-1)(n-2).$$

10·10. General method of solution. When the condition of integrability is satisfied, the integral is found as in the case of three variables. We integrate by keeping all other variables excepting two as constants. The constant of integration is taken to be a function of the variables which were kept constant. We then form the differential of the integral relation obtained. Comparing it with the given equation we can get a relation free of the two variables (which were at first made variable) and their differentials, taking the help of the integral relation if necessary. We thus obtain an equation involving $(n-1)$ differentials instead of n. The rule is reapplied to this and the number is again decreased by unity, and so on, until the solution is obtained.

10·11. If the equation is homogeneous. If the equation is homogeneous, we may put $x=ut$, $y=vt$, $z=wt$ and then integrate, or we can use the method of integrating factor.

The method of procedure will be clear by solved example 14 given below.

Ex. 13. *Solve* : $(2x+y^2+2xz)\,dx+2xy\,dy+x^2\,dz=du.$ (*Agra* 1955)

We have

$$2x\,dx+(y^2\,dx+2xy\,dy)+(2xz\,dx+x^2\,dz)-du=0$$

Integrating $\qquad x^2+y^2x+x^2z-u=c.$

Ex. 14. *Solve* :

$$z(y+z)\,dx+z(u-x)\,dy+y(x-u)\,dz+y(y+z)\,du=0.$$

Here $P=z(y+z)$, $Q=z(u-x)$, $R=y(x-u)$, $T=y(y+z)$. All the conditions of integrability are satisfied There are 4 variables in this equation.

Let us take any two of them as constant. Here for convenience put $y=$constant and $z=$constant so that $dy=0$, $dz=0$. Now the equation becomes

$$z(y+z)\,dx+y(y+z)\,du=0.$$

Integrating $\quad z(y+z)\,x+y(y+z)\,u=\text{constant}=f \qquad (1)$

where f is a function of y and z.

Differentiating it and comparing it with the original equation we get

$$(2zx+2uy)\,dy+(2zx+2uy)\,dz=df$$

or
$$2(zx+uy)(dy+dz)=df$$

or
$$\frac{2f\,(dy+dz)}{y+z}=df \qquad \text{from (1)}$$

Integrating
$$\log(y+z)^2=\log f+\text{constant}$$
or
$$f=c\,(y+z)^2.$$

Hence the complete solution is

$$zx(y+z)+y(y+z)\,u=c\,(y+z)^2$$

i.e.
$$zx+yu=c\,(y+z)$$

Aliter. $Px+Qy+Rz+Su=xz(y+z)+zy(u-x)+yz(x-u)$
$+uy(y+z)=(y+z)(xz+uy)=D$, say

$$dD=(xz+uy)(dy+dz)+(y+z)(xdz+zdx+udy+ydu)$$

Dividing the given equation by D, we easily get

$$\frac{d(D)}{D}-\frac{2(uy+xz)(dy+dz)}{(y+z)(xz+uy)}=0$$

or
$$\frac{d(D)}{D}-2\,\frac{dy+dz}{y+z}=0$$

Integrating $\quad D=c\,(y+z)^2$

$\therefore \quad xz+uy=c\,(y+z)$

10·12. Equations of a degree higher than the first. Sometimes an equation of the form

$$A\,dx^2+B\,dy^2+C\,dz^2+2D\,dy\,dz+2E\,dx\,dz+2F\,dx\,dy=0 \quad \ldots(1)$$

where A, B, C, D, E, F are functions of x, y, z can easily be resolved into factors and the factors can be put in the form

$$Pdx+Qdy+Rdz=0 \text{ and } P'\,dx+Q'\,dy+R'\,dz=0.$$

The solution of either of these, obtained by previous methods, will be a particular solution of the differential equation considered and the two general solutions taken together will form the complete solution.

Now making the use of perfect squares it can easily be found that the equation (1) is resolvable into factors if

$$ABC+2DEF-AD^2-BE^2-CF^2=0.$$

Ex. 15. *Solve :* $x^2\,dx^2+y^2\,dy^2-z^2\,dz^2+2xy\,dx\,dy=0$

This equation satisfies the above condition and is therefore resolvable into factors. These factors give

$$xdx+ydy+zdz=0 \quad \text{whence} \quad x^2+y^2+z^2=a^2$$

and $$xdx+ydy=zdz=0 \quad \text{whence} \quad x^2+y^2-z^2=b^2.$$

Hence the complete solution is

$$(xz+y^2+z^2-a^2)\,(x^2+y^2-z^2-b^2)=0.$$

EXERCISE X (C)

1. $t\,(y+z)\,dx+t\,(y+z+1)\,dy+tdz-(y+z)\,dt=0.$
2. $(2x+y^2+2xt-z)\,dx+2xy\,dy-xdz+x^2\,dt=0.$
3. $dx\,dy\,dz=0.$
4. $(xdx+ydy+zdz)^2\,z=(z^2-x^2-y^2)\,(xdx+ydy+zdz)\,dz.$
5. $ll'\,dx^2+mm'\,dy^2+nn'\,dz^2+(lm'+l'm)\,dx\,dy$
$\qquad +(ln'+l'n)\,dx\,dz+(mn'+m'n)\,dz\,dy=0.$
6. Show that the equation

$$adx^2+bdy^2+cdz^2+2f\,dy\,dz+2g\,dz\,dx+2h\,dx\,dy=0$$

reduces to two equations of the form

$$Pdx+Qdy+Rdz=0$$

if $abc+2fgh-af^2-bg^2-ch^2=0.$

Hence show that the solution of

$$xyz\,(dx^2+dy^2+dz^2)+x\,(y^2+z^2)\,dy\,dz+y\,(z^2+x^2)\,dz\,dx +z\,(x^2+y^2)\,dx\,dy=0$$

is $(x^2+y^2+z^2-c)\,(xyz-c)=0.$

10·13. Non-integrable single differential equations. Let us now consider the equations for which the condition of integrability is not satisfied. Let

$$P\,dx+Q\,dy+R\,dz=0 \tag{1}$$

be an equation for which there exists no single relation between x, y, z to satisfy it.

Let us now assume some integral relation

$$f\,(x, y, z)=0 \tag{2}$$

which on being differentiated gives

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0. \tag{3}$$

Now equations (2) and (3), when the form $f(x, y, z)$ is specified, enable us to eliminate one of the variables and its differential *e. g.* z and dz, from equation (1) and it becomes of the form

$$M\,dx+N\,dy=0,$$

where M and N are functions of x and y. Solving this, we obtain an equation involving an arbitrary constant, and this equation together with (2) will constitute a solution. Sometimes use of relation (2) makes equation (1) integrable.

Note 1 :—*Here every possible solution can be obtained by giving different forms to $f(x, y, z)$.*

Note 2 :—*Such solutions thus found represent the family of curves satisfying (1) that lie on the surfaces represented by (2).*

Ex. 16. *Solve* : $dz=my\,dx+n\,dy$.

Here the condition of integrability is not satisfied. So let us assume some relation between x, y, z and let it be

$$y=f(x),$$

so that $$dy=f'(x)\,dx.$$

Substituting these values in the given equation we get

$$dz=m\,f(x)\,dx+n\,f'(x)\,dx$$

which on integration gives

$$z=m\int f(x)\,dx+nf(x)+c$$

(1) and (2) together give the solution of the equation.

Ex. 17. *Find the most general solution of the equation*

$$x\,dx+y\,dy+c\sqrt{1-x^2/a^2-y^2/b^2}\,dz=0 \tag{1}$$

which is consistent with the assumption that it shall represent a series of lines traced upon the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{2}$$

Equation (1) does not satisfy the condition of integrability. With the help of (2) equation (1) becomes

$$xdx+ydy+zdz=0$$

or $$x^2+y^2+z^2=k^2. \tag{3}$$

Thus (2) and (3) together constitute the solution.

EXERCISE X (D)

1. Show that the curves represented by the solution of

$$y\,dx+z\,dy-y\,dy+x\,dz=0$$

which lie in the plane $2x-y-z=1$ are

$$x\,y+x^2-y^2-y=c^2.$$

2. Show that the curve of

$$x\,dx+y\,dy+c\sqrt{(1-x^2/a^2-y^2/b^2)}\,dz=0$$

that lie on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$

lie also on the family of concentric spheres

$$x^2+y^2+z^2=k^2.$$

3. Show that the curves of $dz=2y\,dx+xdy$ which lie in the plane $z=x+y$ lie on the surfaces of family

$$(x-1)^2\,(2y-1)=c.$$

4. Show that the equation

$$(1+2m)\,x\,dx+y\,(1-x)\,dy+z\,dz=0$$

is not integrable. Further show that any system of lines described on the surface of the sphere

$$x^2+y^2+z^2=r^2,$$

and satisfying the above equation, would be projected on the plane xy in parabolas.

CHAPTER XI

PARTIAL DIFFERENTIAL EQUATIONS OF THF FIRST ORDER

11·1. Introduction. So far we have considered differential equations in which there is only one independent variable. Now we shall consider equations involving two or more independent variables. When we consider the case of two independent variables we shall usually take them to be x and y and take z to be the dependent variable. We shall adopt the notation

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{dy} \text{ also } r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x \partial y},\ t=\frac{\partial^2 z}{\partial y^2};$$

The following facts from Geometry will be made use of in what follows.

(*i*) The direction cosines of normals to a surface are proportional to $p, q, -1$.

(*ii*) Envelope of the surface given by $f(x, y, z, a, b)=0$ is obtained by eliminating a and b from this relation and

$$\frac{\partial f}{\partial a}=0, \frac{\partial f}{\partial b}=0.$$

If we put $b=\phi(a)$ so that the equation becomes

$$f[x, y, z, a, \phi(a)]=0. \tag{1}$$

The envelope then is obtained by eliminating a between (1) and $\frac{\partial f}{\partial a}=0$, where f stands for

$$f[x, y, z, a, \phi(a)].$$

Now we proceed to show that partial differential equations are formed by the elimination of arbitrary constants or arbitrary functions.

11·2. Derivation of a partial differential equation by the elimination of arbitrary constants.

Suppose we have a function

$$f(x, y, z, a, b)=0, \tag{1}$$

where a and b are arbitrary constants.

Regarding x and y as independent variables we get on differentiating

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p=0,\ \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q=0.$$

We have thus three relations from which two quantities a and b can generally be eliminated. The resulting equation

$$F(x, y, z, p, q)=0 \tag{2}$$

is a partial differential equation of the first order.

If there are more arbitrary constants than the number of independent variables the eliminant will give a differential equation of higher order than the first.

Ex. 1. *Find the partial differential equation by the elimination of a and b from*

$$z=ax+by+ab.$$

Differentiating with respect to x and y we get

$$p=a,\ q=b.$$

Substituting these values of a and b in the given equation

$$z=px+qy+pq.$$

Ex. 2. *Form a partial differential equation by eliminating a, b, c from*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Differentiating with respect to x and y

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0,\ \frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\therefore\quad c^2x+a^2z\frac{\partial z}{\partial x}=0, \tag{1}$$

$$c^2y+b^2z\frac{\partial z}{\partial y}=0 \tag{2}$$

Differentiating (1) with respect to x and (2) with respect to y

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \tag{3}$$

$$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \tag{4}$$

(1) and (3) give

$$-z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+xz\frac{\partial^2 z}{\partial x^2}=0.$$

Similarly (2) and (4) give

$$-z\frac{\partial z}{\partial y}+y\left(\frac{\partial z}{\partial y}\right)^2+yz\frac{\partial^2 z}{\partial y^2}=0.$$

EXERCISE XI (A)

Eliminate the arbitrary constants from the following equations and form partial differential equations :

1. $az+b=a^2x+y$.
2. $z=ax+a^2y^2+b$.
3. $z=(x+a)(y+b)$.
4. $(x-h)^2+(y-k)^2+z^2=a^2$.
5. $z=ax\,e^y+\frac{1}{2}a^2e^{2y}+b$.

11·3. Derivation of partial differential equation by the elimination of arbitrary functions.

Suppose u, v two known functions of x, y, z are connected by relation

$$\phi(u, v)=0 \tag{1}$$

where ϕ is arbitrary.

Differentiating (1) with respect to x and y we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0,$$

$$i.e.\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0,$$

$$\text{and } \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial z}+q\frac{\partial v}{\partial z}\right)=0.$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ we obtain

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{dy}+q\frac{\partial u}{\partial z}\right)$$

This may be written as

$$P\,p+Q\,q=R, \tag{2}$$

where $$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=\frac{\partial(u, v)}{\partial(y, z)},$$

$$Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial\,(u,v)}{\partial\,(z,x)},$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial\,(u,v)}{\partial\,(x,y)}:$$

Thus from relation (1) which involves an arbitrary function ϕ we obtain (2), a partial differential equation which is linear that is of first degree in p and q.

If the given relation between x, y, z contains two arbitrary functions then leaving a few exceptional cases the partial differential equations of higher order than the second will be formed.

Ex. 3. *Form the partial differential equation by the elimination of ϕ from*

$$lx+my+nz=\phi\,(x^2+y^2+z^2).$$

Differentiating with respect to x and y we get

$$l+np=\phi'\,(x^2+y^2+z^2)\,.\,(2x+2zp)$$
$$m+nq=\phi'\,(x^2+y^2+z^2)\,.\,(2y+2zq)$$

$$\therefore\quad \frac{l+np}{m+nq}=\frac{x+zp}{y+zq}$$

$$\therefore\quad (l+np)\,y+z\,(lq-mp)=(m+nq)\,x,$$

Ex. 4. *Eliminate the arbitrary functions from the equation*

$$z=f\,(x+iy)+F\,(x-iy).$$

$$\frac{\partial z}{\partial x}=f'\,(x+iy)+F'\,(x-iy)$$

$$\frac{\partial^2 z}{\partial x^2}=f''\,(x+iy)+F''\,(x-iy)$$

$$\frac{\partial z}{\partial y}=i\,f'\,(x+iy)-i\,F'\,(x-iy)$$

$$\frac{\partial^2 z}{\partial y^2}=-f''\,(x+iy)-F''\,(x-iy)$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2}=-\frac{\partial^2 z}{\partial y^2}.$$

EXERCISE XI (B)

Eliminate the arbitrary functions from the following equations :

1. $z=e^{ax+by}\,f\,(ax-by)$ **2.** $z=F\,(x^2+y^2)$.

3. $z=f\left(\frac{y}{x}\right)$ 4. $z=y^2+2f\left(\frac{1}{x}+\log y\right)$.

5. $z=f(x+ay)+\phi(x-ay)$.

11·5. Classification of Solutions. As in the case of ordinary differential equations, a relation between the variables is a *solution* of the given partial differential equation provided that on solving for the dependent variable and substituting into the given equation, it is satisfied *i.e.* it is reduced to an identity in the independent variables.

An ordinary differential equation has solutions involving a few arbitrary constants while a partial differential equation has solutions involving arbitrary functions so that by particularizing the function any number of arbitrary constants can be inserted. Thus we see that a partial differential equation is richer in solutions than an ordinary differential equation.

Complete Integral. We have seen that an elimination of two arbitrary constants leads to a partial differential equation of first order in the case when there are two independent variables.

A relation between the variables which satisfies the equation and which contains as many arbitrary constants as there are independent variables is known as *Complete Integral* of the given partial differential equation.

Particular Integral. By giving particular values to the arbitrary constants of the complete integral we get a solution of the partial differential equation and this solution is called a *Particular Integral* of the given partial differential equation.

General Integral. We have seen in Art. 11·3 that elimination of an arbitrary function from the relation $\phi(u, v)=0$, where u and v are two independent functions of x, y, z leads to a partial differential equation of first order. Hence when a partial differential equation of first order is given it will have a solution $\phi(u, v)=0$. This is called the General Integral of the derived equation.

Thus the *General Integral* of a partial differential equation of first order in the case of two independent variables is a relation between the variables involving two independent functions of those variables together with an arbitrary function of those two functions. In the case of n independent variables it is a relation between the variables involving $n-1$ independent functions of those variables together with an arbitrary function of those $n-1$ functions.

Singular Solution. The equation of the envelope of the surfaces represented by the Complete Integral is found. This equation of the envelope is called the *Singular Integral* of the

$$Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial (u, v)}{\partial (z, x)},$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial (u, v)}{\partial (x, y)};$$

Thus from relation (1) which involves an arbitrary function ϕ we obtain (2), a partial differential equation which is linear that is of first degree in p and q.

If the given relation between x, y, z contains two arbitrary functions then leaving a few exceptional cases the partial differential equations of higher order than the second will be formed.

Ex. 3. *Form the partial differential equation by the elimination of ϕ from*

$$lx+my+nz=\phi\ (x^2+y^2+z^2).$$

Differentiating with respect to x and y we get

$$l+np=\phi'\ (x^2+y^2+z^2)\ .\ (2x+2zp)$$
$$m+nq=\phi'\ (x^2+y^2+z^2)\ .\ (2y+2zq)$$

$$\therefore\quad \frac{l+np}{m+nq}=\frac{x+zp}{y+zq}$$

$$\therefore\quad (l+np)\ y+z\ (lq-mp)=(m+nq)\ x,$$

Ex. 4. *Eliminate the arbitrary functions from the equation*

$$z=f\ (x+iy)+F\ (x-iy).$$

$$\frac{\partial z}{\partial x}=f'\ (x+iy)+F'\ (x-iy)$$

$$\frac{\partial^2 z}{\partial x^2}=f''\ (x+iy)+F''\ (x-iy)$$

$$\frac{\partial z}{\partial y}=i f'\ (x+iy)-i F'\ (x-iy)$$

$$\frac{\partial^2 z}{\partial y^2}=-f''\ (x+iy)-F''\ (x-iy)$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2}=-\frac{\partial^2 z}{\partial y^2}.$$

EXERCISE XI (B)

Eliminate the arbitrary functions from the following equations :

1. $z=e^{ax+by} f\ (ax-by)$ 2. $z=F\ (x^2+y^2)$.

As a result of this elimination of p and q we obtain a relation between x, y, z which is independent of any arbitrary constants. This relation will be the Singular Integral *if it satisfies the given differential equation.*

11·7. General Integral from Complete Integral. The Complete Integral can be used to derive an integral involving an arbitrary function. Let the Complete Integral be

$$F(x, y, z, a, b)=0, \tag{1}$$

and let one of the constants be a function of the other *i. e.* let $b=\psi(a)$, then the Complete Integral becomes

$$F[x, y, z, a, \psi(a)]=0. \tag{2}$$

The General Integral is obtained by eliminating the constant a between (2) and

$$\frac{\partial F}{\partial a}=0. \tag{3}$$

The equations (2) and (3), with a retained, obviously represent the curve of intersection of two consecutive surfaces of the system $F[x, y, z, a, \psi(a)]=0$. The envelope of the family of surfaces is the locus of the intersections of consecutive surfaces and hence contains the curve. This curve is called the *characteristic* of the envelope.

Thus we may also define General Integral as the locus of the characteristics.

While deriving General Integrals from the Complete Integrals sometimes other relations also appear. To confirm that the relations give General Integral, we must as in the case of singular integrals, verify that they satisfy the given differential equation.

It may be noted that as by the elimination of an arbitrary function we always get a linear equation, the general integral of a non-linear equation can not be expressed by a single relation. Important cases can arise when we give particular values to the functions.

11·8. Solutions of a partial differential equation. If we want to solve a given partial differential equation, we must not only find the Complete Integral but should also indicate the Singular and General Integrals. If we simply find out the Complete Integral and do not indicate the Singular and General Integrals, partial differential equation is not considered as completely solved.

We will see however that Lagranges Method of solving a partial differential equation of the first order enables us to obtain an integral involving an arbitrary function. It can be shown that this provides all solutions of the equation which are not of the type

known as 'special solutions' [See Art 11·10]. Hence the partial differential equation of the 1st order and first degree is considered as completely solved if the solution is obtained by Lagrange's Method in the form $\phi\ (u, v)=0$.

11·9. Lagrange's Linear Equation. This is the name given to equations of the form

$$Pp+Qq=R, \tag{1}$$

where P, Q, R are functions of x, y, z.

We have seen in Art. 11·3 that by eliminating an arbitrary function ϕ from

$$\phi\ (u, v)=0 \tag{2}$$

*we form the partial differential equation

$$Pp+Qq=R, \tag{3}$$

where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z},\ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Hence when we have an equation given by (3) we have an integral given by (2) which is called its General Integral. The main problem is therefore to find u and v for insertion in the equation (2).

Let us consider equations $u=a$ and $v=b$ in which a and b are arbitrary constants and form the differential equations corresponding to them.

We have on differentiation

$$\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy+\frac{\partial u}{\partial z}\,dz=0$$

and

$$\frac{\partial v}{\partial x}\,dx+\frac{\partial v}{\partial y}\,dy+\frac{\partial v}{\partial z}\,dz=0,$$

which give on solving

$$\frac{dx}{\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial v}{\partial y}}=\frac{dy}{\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial z}}=\frac{dz}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x}}$$

$$i.\,e.\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}, \tag{4}$$

Now (4) are differential equations whose solutions are $u=a$ and $v=b$.

Hence we have the following rule for solving a linear partial differential equation :—

*For complete proof this must be proved.

To obtain a solution of the partial differential equation

$$Pp+Qq=R,$$

write down the subsidiary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R};$$

find two independent integrals of the subsidiary equations, say, u=constant, v=constant ; then the General Integral of the partial differential equation is given by

$$\phi\ (u, v)=0$$

were ϕ is an arbitrary function.

It may be noted that an arbitrary functional relation between u and v of any form will be found satisfactory. Thus we can also have the General Integral in the form

$$u=\psi\ (v),$$

where ψ is an arbitrary function.

The method can be easily generalised. Thus to find an integral of

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+\ldots\ldots+P_n\frac{\partial z}{\partial x_n}=0,$$

we form the subsidiary equations

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\frac{dx_3}{P_3}=\ldots\ldots=\frac{dx_n}{P_n}$$

and let $u_1=a_1$, $u_2=a_2,\ldots\ldots\ldots$, $u_n=a_n$ be n independent integrals of these, then the General Integral of the given partial differential equation is

$$F\ (u_1, u_2,\ldots\ldots u_n)=0,$$

where F is an arbitrary function.

11·10. Special Integral. It is not always that all integrals of Lagrange's linear equation are included in $\phi\ (u, v)=0$. Let us take the equation

$$p+2qz^{1/3}=3z^{2/3},$$

whose subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{2z^{1/3}}=\frac{dz}{3z^{2/3}},$$

giving $x-z^{1/3}$=constant and $y-z^{2/3}$=constant as two independent solutions. Hence General Integral is

$$\phi\ (x-z^{1/3}, y-z^{2/3})=0.$$

However, we see that $z=0$ satisfies the given partial differential equation. Clearly it is impossible to express it as a function of $x-z^{1/3}$ and $y-z^{2/3}$ [*i e.* u and v].

An integral of this type is known as *special integral.*

EXERCISE XI (C)

Show that the following equations possess the given special integrals.

1. $p+q\,[\,1+(z-y)^{1/3}\,]=1\;;\;z=y.$
2. $p-q=2\sqrt{z}\;;\;z=0.$
3. $[\,1+\sqrt{z-x-y}\,]\,p+q=2\;;\;z=x+y.$

11·11. Geometrical Interpretation of Pp+Qq=R.

We can write the equation as

$$Pp+Qq+R\,(-1)=0. \tag{1}$$

Now the direction cosines of the normal to a surface at a point are proportional to $p:q:-1$ [If the surface is $f(x, y, z)=0$, *d. c's* are proportional to

$$\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{dz} \quad \text{or} \quad -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial z},\ -\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial z},\ -1$$

or $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, -1$ *i. e.* $p, q, -1$]. Hence the geometrical interpretation of (1) is that the normal at a point to a certain surface is perpendicular to a line whose direction cosines are in the ratio of $P:Q:R$.

We know that the simultaneous equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{2}$$

represent a family of curves such that the tangent at any point has direction cosines proportional to $P:Q:R$. Also that if $u=$constant and $v=$constant are two particular integrals of these, then $\phi\,(u, v)=0$ represents a surface through these curves. Take any point on this surface, then through this point must pass a curve of the family which lies entirely on the surface. The normal to this surface at the point taken must, therefore, be at right angles to the tangent at this point to the curve *i. e.* it is perpendicular to a line whose direction cosines are proportional to $P:Q:R$; and this is exactly what is required by the partial differential equation (1).

Hence equations (1) and (2) define the same set of surfaces and are thus equivalent.

Ex. 5. *Solve* : $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$. (*Agra* 1939)

The subsidiary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}$$

Each of these ratios$=\dfrac{x\,dx+y\,dy-dz}{0}=\dfrac{\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}}{0}$

Hence we get $x^2+y^2-2z=a$ and $xyz=b$

$\therefore$ General solution is

$$\phi(x^2+y^2-2z,\ xyz)=0.$$

Ex. 6. *Solve* : $(z^2-2yz-y^2)p+(xy+xz)q=xy-xz$. (*Agra* '49)

The subsidiary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$$

or $$\frac{xdx}{z^2-2yz-y^2}=\frac{dy}{y+z}=\frac{dz}{y-z}=\frac{xdx+ydy+zdz}{0}$$

$$\therefore\quad x^2+y^2+z^2=a$$

Taking 2nd and 3rd terms we get

$$ydy-zdy-ydz-zdz=0$$
$$y^2-2yz-z^2=b.$$

The General Integral is

$$\phi(x^2+y^2+z^2,\ y^2-2yz-z^2)=0$$

Ex. 7. *Solve* : $p+3q=5z+\tan(y-3x)$.

The subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$$

Integrating 1st two terms, we get $y=3x+a$.

Taking 1st and 3rd terms

$$\frac{dx}{1}=\frac{dz}{5z+\tan a}\text{, writing } a \text{ for } (y-3x)$$

or $$\frac{dz}{dx}-5z=\tan a$$

$$\therefore\quad z\,e^{-5x}=\tan a\,\frac{e^{-5x}}{(-5)}+b.$$

The General Integral is

$$\phi\left[\,y-3x,\ e^{-5x}\{\,5z+\tan(y-3x)\,\}\,\right]=0.$$

Ex. 8. *Solve* : $z-xp-yq=a\sqrt{x^2+y^2+z^2}$.

(*Raj.* '49, *Nagpur* 1961)

The eq^n is $xp+yq=z-a\sqrt{x^2+y^2+z^2}$.

The subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{x^2+y^2+z^2}}=\frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2-az\sqrt{x^2+y^2+z^2}}$$

$$=\frac{u\,du}{u^2-az\,u}=\frac{du}{u-az}=\frac{dz}{z-au}\text{ , putting } u^2=x^2+y^2+z^2$$

$$=\frac{du+dz}{(1-a)(u+z)}$$

First two terms give $y=c_1x$.

Taking 1st and last terms we get.

$$c_2x^{1-a}=u+z$$

$$\therefore\quad z+(x^2+y^2+z^2)^{1/2}=c_2x^{1-a}.$$

The General Integral is

$$z+\sqrt{x^2+y^2+z^2}=x^{1-a}\phi\left(\frac{y}{x}\right)$$

Ex. 9. *Solve* :

$$(2x^2+y^2+z^2-2yz-zx-xy)\,p+(x^2+2y^2+z^2-yz-2zx-xy)\,q$$
$$=x^2+y^2+2z^2-yz-zx-2xy$$

(*Agra* 1934)

We have

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}$$

$$=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}$$

Then $$\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}$$

i.e. $$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

$$\therefore\quad\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

i.e. $\log\dfrac{x-y}{y-z}=\text{const}$ *i.e.* $\dfrac{x-y}{y-z}=\text{const}.$

Similarly $\dfrac{y-z}{z-x}=\text{const.}$

$\therefore$ Solution is $\phi\left(\dfrac{x-y}{y-z}, \dfrac{y-z}{z-x}\right)$

Ex. 10. *Solve* : $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}+t\dfrac{\partial z}{\partial t}=az+\dfrac{xy}{t}$. (*Agra* 1962)

The subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{az+(xy/t)}.$$

Integrating the equation formed by first and second terms,

$$y=c_1 x.$$

Similarly from 1st and 3rd terms $t=c_2 x$.

Taking 1st and 4th terms we have

$$\frac{dx}{x}=\frac{dz}{az+x.\,c_1/c_2} \quad \text{or} \quad \frac{dz}{dx}-a\frac{z}{x}=\frac{c_1}{c_2}.$$

The integrating factor is $e^{-\int (a/x)\,dx}=\dfrac{1}{x^a}$

$$\therefore \quad z.\frac{1}{x^a}=\frac{c_1}{c_2}\frac{x^{1-a}}{(1-a)}+c_3$$

$$\therefore \quad \frac{z}{x^a}-\frac{y}{t}\frac{x^{1-a}}{(1-a)}=c_3, \text{ since } \frac{c_1}{c_2}=\frac{y}{t};$$

The General Integral is

$$\frac{z}{x^a}+\frac{y}{t}\frac{x^{1-a}}{(a-1)}=\phi\left(\frac{y}{x}, \frac{t}{x}\right)$$

i.e. $\quad (a-1)\,z+\dfrac{dy}{t}=x^a\,(a-1)\,\phi\left(\dfrac{y}{x}, \dfrac{t}{x}\right)$

$$=x^a\,\psi\left(\frac{y}{x}, \frac{t}{x}\right).$$

Ex. 11. *Solve* :

$(x_2+x_3+z)\,p_1+(x_3+x_1+z)\,p_2+(x_1+x_2+z)\,p_3=x_1+x_2+x_3.$

The subsidiary equations are

$$\frac{dx_1}{x_2+x_3+z}=\frac{dx_2}{x_3+x_1+z}=\frac{dx_3}{x_1+x_2+z}=\frac{dz}{x_1+x_2+x_3}$$

$$=\frac{dz-dx_1}{-(z-x_1)}=\frac{dz-dx_2}{-(z-x_2)}=\frac{dz-dx_3}{-(z-x_3)}$$

$$=\frac{dx_1+dx_2+dx_3+dz}{3\,(x_1+x_2+x_3+z)}.$$

Hence we get from the 5th and last term

$$\log c-\log (z-x_1)=\log (x_1+x_2+x_3+z)^{1/3}$$

$$\therefore\quad \frac{c_1}{z-x_1}=(x_1+x_2+x_3+z)^{1/3}=T^{1/3}.\ \text{say}$$

Similarly we obtain

$$\frac{c_2}{z-x_2}=T^{1/3},\ \frac{c_3}{z-x_3}=T^{1/3}$$

Therefore the General Integral is

$$\phi\,[\,(z-x_1)\,T^{1/3},\,(z-x_2)\,T^{1/3},\,(z-x_3)\,T^{1/3}\,]=0,$$

where T stands for $x_1+x_2+x_3+z$.

EXERCISE XI (D)

Solve the equations :

1. $yzp+zxq=xy$.
2. $(y+z)\,p+(z+x)\,q=x+y$. (*Agra* '55 '61)
3. $zp-zq=z^2+(y+x)^2$ (*Luck.* '54)
4. $xzp+yzq=xy$.
5. $x^2\,y+y^2\,q=nxy$.
6. $p-q=\dfrac{z}{x+z}$.
7. $(y+x)\,p+(y-x)\,q=z$.
8. $p\tan x+q\tan y=\tan z$.
9. $xy^2\,p-y^3\,q+axz=0$.
10. $(3x+y-z)\,p+(x+y-z)\,q=2\,(z-v)$.
11. $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=xyz$.
12. $x_2\,x_3\,z\,p_1+x_3\,x_1\,z\,p_2+x_1\,x_2\,z\,p_3=x_1\,x_2\,x_3$.

11·12. Special Methods of Solution. We have given the method of solution of the linear equation $Pp+Qq=R$. There is a general method for solving partial differential equations of first order *but of any degree*. However, before giving this method we will give certain types of equations whose solutions are quite easily obtained. Many equations can be reduced to a few *standard forms* and hence oan be solved generally by methods which are shorter than the general method.

11·13. Standard I. Equations of the form f (p, q) = 0.

There are some equations in which x, y and z do not occur explicitly and they can be written as

$$f(p, q)=0. \tag{1}$$

The Complete Integral is given by

$$z=ax+by+c, \tag{2}$$

where a and b are connected by the ralation

$$f(a, b)=0,$$

since from (2) $p=\dfrac{\partial z}{\partial x}=a$ and $q=\dfrac{\partial z}{\partial y}=b$ which when substituted in(3) give (1).

General Integral can be calculated by the usual method and it can be easily shown that equations of the type given by (1) have no Singular Integral [Since when we differentiate (2) with respect to c we get $1=0$].

Ex. 12. *Solve :* $p^2+q^2=m^2$.

This is an equation of standard form 1 hence its Complete Integral is given by

$$z=ax+by+c, \quad \text{where} \quad a^2+b^2=m^2.$$

$\therefore$ Complete Integral is $z=ax+\sqrt{m^2-a^2}\,.\,y+c$

The General Integral is obtained by eliminating a between the equations

$$z=ax+\sqrt{m^2-a^2}\,.\,y+\phi(a), \text{ writing } c=\phi(a)$$

$$\text{and} \quad 0=x-\frac{a}{\sqrt{m^2-a^2}}\,.\,y+\phi'(a)$$

[It is easily seen that Singular Integral does not exist].

Cor. Sometimes change of variables enables us to bring equations to the form of Standard I. This can be understood with the help of the solved examples given below

Ex. 13. *Solve :* $x^2p^2+y^2q^2=z^2$.

The given equation is

$$\left(\frac{z^{-1}}{x^{-1}}\frac{\partial z}{\partial x}\right)^2+\left(\frac{z^{-1}}{y^{-1}}\frac{\partial z}{\partial y}\right)^2=1$$

If $dZ=z^{-1}\,dz$ so that $Z=\log z$,
$dX=x^{-1}\,dx$ so that $X=\log x$,
and $dY=y^{-1}\,dy$ so that $Y=\log y$,

then the equation becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=1,$$

which can be solved as in Example 12 to give the Complete Integral $Z=aX+\sqrt{1-a^2}\,.\,Y+c_1$

$\therefore$ $\log z=\cos\alpha\log x+\sin\alpha\log y+\log c$, putting $a=\cos\alpha$,

$$\therefore\quad z=c\,x^{\cos\alpha}y^{\sin\alpha}.$$

For *S. I.* we have

$$x^{\cos\alpha}y^{\sin\alpha}=0,$$

and $\quad -c\sin\alpha\,x^{\cos\alpha}y^{\sin\alpha}+c\cos\alpha\,x^{\cos\alpha}y^{\sin\alpha}=0$

$\therefore$ $z=0$ is the Singular Integral.

General Integral is calculated by eliminating α betwen

$$z=\phi(\alpha)\,x^{\cos\alpha}y^{\sin\alpha}\text{ and }0=\phi'(\alpha)\,x^{\cos\alpha}y^{\sin\alpha}$$

$$+\phi(\alpha)\,y^{\sin\alpha}x^{\cos\alpha}(-\sin\alpha)+\phi(\alpha)\,x^{\cos\alpha}y^{\sin\alpha}\cos\alpha,$$

where $c=\phi(\alpha)$ which is an arbitrary function of α.

Ex. 14. *Find the Complete Integral of*

$$(y-x)(qy-px)=(p-q)^2 \qquad (\textit{Agra } 1956)$$

Put $u=x+y$, $v=xy$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}.$$

The given equation becomes

$$(y-x)\left[\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)y-\left(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)x\right]$$

$$=\left[\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}-x\frac{\partial z}{\partial v}\right]^2$$

$$\therefore\quad (y-x)^2\frac{\partial z}{\partial u}=(y-x)^2\left(\frac{\partial z}{\partial v}\right)^2\text{ or }\frac{\partial z}{\partial u}=\left(\frac{\partial z}{\partial v}\right)^2$$

The Complete Integral is

$$z=au+bv+c,\text{ where }a=b^2$$

$$\therefore\quad z=a(x+y)+\sqrt{a}\,xy+c,$$

Ex. 15. *Solve* : $(x+y)(p+q)^2+(x-y)(p-q)^2=1.$ (*Agra* 1958)

Put $\sqrt{x+y}=X,\ \sqrt{x-y}=Y$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$=\frac{\partial z}{\partial X}\cdot\frac{1}{2\sqrt{x+y}}+\frac{\partial z}{\partial Y}\cdot\frac{1}{2\sqrt{x-y}}$$

$$=\tfrac{1}{2}\left(\frac{\partial z}{X\partial X}+\frac{\partial z}{Y\partial Y}\right)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}+\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y}=\frac{1}{2}\left(\frac{\partial z}{X\partial X}-\frac{\partial z}{Y\partial Y}\right)$$

$$\therefore\quad p+q=\frac{\partial z}{X\partial X};\ p-q=\frac{\partial z}{Y\partial Y}$$

Substituting these values the given equation becomes

$$\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2=1$$

$\therefore$ The complete integral is $z=aX+\sqrt{1-a^2}\,Y+c$

$$=a\sqrt{x+y}+\sqrt{1-a^2}\,\sqrt{x-y}+c.$$

Ex. 16. *Solve* : $(x^2+y^2)(p^2+q^2)=1.$ (*Luck.* 1956)

Put $x=r\cos\theta,\ y=r\sin\theta$ *i.e.* $r^2=x^2+y^2$ and $\theta=\tan^{-1}y/x.$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial z}{\partial r}\cos\theta-\frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}$$

also $q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial r}\sin\theta+\dfrac{\partial z}{\partial \theta}\dfrac{\cos\theta}{r}$.

Substituting these values the given equation becomes

$$r^2\left(\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1 \qquad \text{...(1)}$$

Putting $d\rho=\dfrac{dr}{r}$ *i.e.* $\rho=\log r$, (1) becomes

$$\left(\frac{\partial z}{\partial \rho}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=1$$

Hence integral $z=a\rho+b\theta+c$, where $b=\sqrt{1-a^2}$

$$=a\log r+\sqrt{1-a^2}\,\theta+c$$

$$=\tfrac{1}{2}a\log(x^2+y^2)+\sqrt{1-a^2}\tan^{-1}y/x+c.$$

Ex. 17. *Solve* : $p^m \sec^{2m} x + z^l q^n \operatorname{cosec}^{2n} y = z^{lm/(m-n)}$.

Substituting $\cos^2 x\, dx = dX$, $\sin^2 y\, dy = dY$, $z^{-l/(m-n)}\, dz = dZ$, and therefore $X = \frac{1}{2}(x + \frac{1}{2}\sin 2x)$, $Y = \frac{1}{2}(y - \frac{1}{2}\sin 2y)$

$$Z = \frac{m-n}{m-n-l}\, z^{(m-n-l)/(m-n)},$$

the given equation becomes

$$\left(\frac{z^{-l/(m-n)}}{\cos^2 x} \cdot \frac{\partial z}{\partial x}\right)^m + \left(\frac{z^{-l/(m-n)}}{\sin^2 x} \cdot \frac{\partial z}{\partial y}\right)^n = 1$$

$$i.e. \quad \left(\frac{dZ}{dX}\right)^m + \left(\frac{dZ}{dY}\right)^n = 1$$

$\therefore$ The complete integral is

$Z = aX + bY + c$ where $a^m + b^n = 1$ and X, Y, Z have the above values.

EXERCISE XI (E)

Solve the equations :

1. $pq = 1$.
2. $p^2 - q^2 = 1$.
3. $pq = p + q$.

11·14. Standard II. Equations of the form

$$f(z, p, q) = 0 \tag{1}$$

Let us assume a tentative solution $z = f(x + ay)$ where a is *an arbitrary constant*

$\therefore \quad z = f(X)$, where $X = x + ay$;

$$\text{then } p = \frac{dz}{dX}\frac{\partial X}{\partial x} = \frac{dz}{dX},$$

$$\text{and } q = \frac{dz}{dX}\frac{\partial X}{\partial y} = a\frac{dz}{dX},$$

Equation (1) becomes

$$f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0,$$

which is an ordinary differential equation of the 1st order. By integrating it we shall get the Complete Integral. General and Singular Integrals are to be found in the usual manner.

Thus to integrate an equation of the form given by (1) put $\frac{dz}{dX}$ for p and a $\frac{dz}{dX}$ for q then solve the equation. After solving it substitute $x + ay$ for X.

Ex. 18. *Solve : Find the Complete Integral of*

$$p^2=z^2(1-pq).$$

Put $z=f(x+ay)=f(X)$, hence we get

$$\left(\frac{dz}{dX}\right)^2=z^2\left[1-a\left(\frac{dz}{dX}\right)^2\right]$$

or $$dX=\frac{\sqrt{1+a z^2}}{z}\,dz=\frac{1+a z^2}{z\sqrt{1+a z^2}}\,dz$$

$$=\frac{1}{z\sqrt{1+a z^2}}\,dz+\frac{az}{\sqrt{1+a z^2}}\,dz$$

$$\therefore\ X+c=\frac{1}{\sqrt{a}}\log\left(z\sqrt{a}+\sqrt{1+a z^2}\right)+\sqrt{1+a z^2}$$

Ex. 19. *Solve : Find the Complete Integral of*

$$q^2 y^2=z(z-px).$$

Putting $\dfrac{dx}{x}=dX$ so that $X=\log x$

and $\dfrac{dy}{y}=dY$ so that $Y=\log y$

the given equation becomes

$$\left(\frac{\partial z}{\partial Y}\right)^2=z\left(z-\frac{\partial z}{\partial X}\right)$$

Let $z=f(X+aY)=f(u)$ where $u=X+aY$

For $\dfrac{\partial z}{\partial X}$ put $\dfrac{dz}{du}$ and for $\dfrac{\partial z}{\partial Y}$ put $a\dfrac{dz}{du}$

$$\therefore\ a^2\left(\frac{dz}{du}\right)^2+z\frac{dz}{du}-z^2=0$$

$$\frac{dz}{du}=\frac{-z\pm\sqrt{z^2+4a^2z^2}}{2a^2}=\frac{z}{2a^2}\left(\pm\sqrt{1+4a^2}-1\right)$$

$$\therefore\ 2a^2\log z=\left(-1\pm\sqrt{1+4a^2}\right)\left[u+\log b\right]$$

$$=\left(\pm\sqrt{1+4a^2}-1\right)\left[(\log x+a\log y)+\log b\right]$$

$$\therefore\ z^{2a^2/(\pm\sqrt{1+4a^2}-1)}=bxy^a$$

which is the required integral.

EXERCISE XI (F)

Solve the equations :

1. $p(1+q^2)=q(z-a)$. (*Agra* 1960)
2. $p^3+q^3=27z$.
3. $z^2(p^2+q^2+1)=c^2$. (*Punjab* 1954)
4. $z=p^m q^m$.
5. $9(p^2 z+q^2)=4$. (*Agra* 1959)
6. $x^2 p^2+y^2 q^2=z^2$ (*Nagpur* 1958)

11·15. Standard III. To this standard belong equations of the form

$$f(x, p)=F(y, q).$$

As a trial solution we assume

$$f(x, p)=F(y, q)=a.$$

From these equations we obtain

$$p=f_1(x, a),\ q=f_2(y, a).$$

Now $dz=p\,dx+q\,dy=f_1(x, a)\,dx+f_2(y, a)\,dy$

$\therefore\ z=\int f_1(x, a)\,dx+\int f_2(y, a)\,dy+b$

The General Integral may be deduced in the usual manner. As in Standard I there is no Singular Integral.

Ex. 20. *Solve* : $\sqrt{p}+\sqrt{q}=2x$. (*Agra* 1956)

We have $\sqrt{p}-2x=-\sqrt{q}=\lambda$.

$\therefore\ p=(\lambda+2x)^2$ and $q=\lambda^2$.

Now $dz=p\,dx+q\,dy=(\lambda+2x)^2\,dx+\lambda^2\,dy$

$\therefore\ z+c=\frac{1}{6}(\lambda+2x)^3+\lambda^2 y$.

Putting $c=\phi(\lambda)$ we get

$$z+\phi(\lambda)=\tfrac{1}{6}(\lambda+2x)^3+\lambda^2 y \qquad (1)$$

differentiating with respect to λ

$$\phi'(\lambda)=\tfrac{1}{2}(\lambda+2x)^2+2\lambda y \qquad (2)$$

Eliminating λ between (1) and (2) we get the General Integral.

Ex. 21. *Solve* : $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=x^2+y^2$.

Putting $z\,dz=dZ$ so that $Z=z^2/2$, the given equation becomes

$$p^2+q^2=x^2+y^2, \text{ where } p=\frac{\partial Z}{\partial x} \text{ and } q=\frac{\partial Z}{\partial y}.$$

$\therefore\ p^2-x^2=y^2-q^2=\lambda$, say

or $p=\sqrt{x^2+\lambda}$ and $q=\sqrt{y^2-\lambda}$.

$\therefore\ dZ=p\,dx+q\,dy=\sqrt{x^2+\lambda}\,.\,dx+\sqrt{y^2-\lambda}\,dy$

or $\frac{1}{2}x\sqrt{x^2+\lambda}+\frac{1}{2}\lambda\log(x+\sqrt{x^2+\lambda})$

$+\frac{1}{2}y\sqrt{y^2-\lambda}-\frac{1}{2}\lambda\log(y+\sqrt{y^2-\lambda})=Z+c.$

$=z^2/2+c.$

The Genegral Integral is found in the usual manner.

Aliter. $z^2 p^2 - x^2 = y^2 - z^2 q^2 = a^2$ (say)

$$\therefore \quad p = \frac{\sqrt{a^2+x^2}}{z}, \; q = \frac{\sqrt{y^2-a^2}}{z}$$

Substituting in $dz = p\,dx + q\,dy$

$$z\,dx = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$$

Integrating

$$z^2 = x\sqrt{x^2+a^2} + a^2 \log\left(x + \sqrt{x^2+a^2}\right) + y\sqrt{y^2-a^2} - a^2 \log\left(y + \sqrt{y^2-a^2}\right)$$

or $$z^2 = x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2 \log \frac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}.$$

EXERCISE XI (G)

Solve the equations :

1. $pq = xy$. (*I. A. S.* 1950 *Raj.* 1961)
2. $p^2 + q^2 = x + y$. (*Agra* 1954)
3. $yp = 2yx + \log q$.
4. $q = xyp^2$.

11·16. Standard IV. In this class are included those partial differential equations which are analogous to Clairaut's form in ordinary differential equations.

In the case of two independent variables they are represented by

$$z = px + qy + f(p, q)$$

The Complete Solution is

$$z = a\,x + by + f(a, b), \tag{1}$$

$$\text{for} \quad p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = b$$

In order to obtain the General Integral put $b = \phi\,(a)$ where ϕ denotes an arbitrary function

$$\therefore \quad z = ax + y\,\phi\,(a) + f\,[a, \phi\,(a)].$$

Differentiating with respect to a.

$$\therefore \quad 0 = x + y\,\phi'\,(a) + f'\,(a),$$

and eliminate a between these equations.

In order to obtain the Singular Integral, differentiate (1) with respect to a and b.

$$\therefore \quad 0 = x + \frac{\partial f}{\partial a}, \tag{2}$$

$$0 = y + \frac{\partial f}{\partial b}. \tag{3}$$

and eliminate a and b between equations (1), (2) and (3).

Ex. 22. *Solve* : $z=px+qy+pq.$

The Complete Integral is

$$z=ax+by+ab. \tag{1}$$

For Singular Integral we differentiate with respect to a and b.

$$\therefore \quad 0=x+b, \tag{2}$$
$$0=y+a. \tag{3}$$

Eliminating a and b between (1), (2) and (3), we get

$$z=-xy-xy+xy=-xy.$$

This is the Singular Integral, as it satisfies the differential equation.

For General Integral, we write $b=\phi\ (a)$.

The equation (1) becomes

$$z=ax+y\ \phi\ (a)+a\ \phi\ (a). \tag{4}$$

Differentiating with respect to a

$$0=x+y\ \phi'\ (a)+\phi\ (a)+a\ \phi'\ (a) \tag{5}$$

The a-eliminant of (4) and (5) is the General Integral.

Ex. 23. *Solve* : $z=px+qy+c\sqrt{1+p^2+q^2}$

(*Agra* 1953, 1962, *Nagpur* 1958)

The Complete Integral is

$$z=ax+by+c\sqrt{1+a^2+b^2}. \tag{1}$$

For the Singular Integral differentiating with respect to a and b we get

$$0=x+\frac{ac}{\sqrt{1+a^2+b^2}}, \tag{2}$$

$$0=y+\frac{bc}{\sqrt{1+a^2+b^2}}. \tag{3}$$

$$\therefore \quad x^2+y^2=\frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

or $\quad c^2-x^2-y^2=\dfrac{c^2}{1+a^2+b^2}$ *i. e.* $1+a^2+b^2=\dfrac{c}{c^2-x^2-y^2}$

$$\therefore \quad a=-\frac{x\sqrt{1+a^2+b^2}}{c}=\frac{-x}{\sqrt{c^2-x^2-y^2}}$$

Similarly $\quad b=\dfrac{-y}{\sqrt{c^2-x^2-y^2}}$

Substituting in (1) we get

$$x^2+y^2+z^2=c^2.$$

EXERCISE XI (H)

Solve the equations :

1. $z=px+qy+\log pq.$

2. $z=px+qy+p^2+q^2.$ (*Punjab* '56 ; *Agra* '59 ; *Nagpur* '57)

3. $z=px+qy-2\sqrt{pq}.$

4. $z=px+qy-n\,p^{1/n}\,q^{1/n}.$

11.17. Charpit's Method. We shall now consider Charpit's method of solving equations *with two independent variables.* As has been mentioned before, this method is to be applied when the given equation can not be reduced to any of the standard forms because the solution by Charpit's method is generally more cumbersome.

Let the equation to be solved be denoted by

$$f(x, y, z, p, q)=0. \tag{1}$$

We have also

$$dz=p\,dx+q\,dy. \tag{2}$$

The essence of Charpit's method consists in finding another relation

$$F(x, y, z, p, q)=0 \tag{3}$$

such that when the values of p and q, derived from it and the given equation (1), are substituted in (2), it becomes integrable. The integral of (2) thus obtained will obviously satisfy (1) for the values of p and q derived from it are no other than those previously obtained from (1).

Assume then that (3) is the relation of the required type.

Therefore we may consider z, p, q expressed as functions of x and y so that when these values are substituted in $f=0$, $F=0$ they are satisfied identically. Therefore their differential coefficients with respect to x and y will vanish. Hence

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0,$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0,$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0,$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0.$$

Eliminating $\frac{\partial p}{\partial x}$ from the 1st pair of equations, we get

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p}-\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial q}p+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p}=0$$

Therefore

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)=0 \tag{4}$$

Similarly eliminating $\frac{\partial q}{\partial y}$ between the last pair of equations we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)=0$$

$$\text{Now } \frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial p}{\partial y}. \tag{5}$$

Therefore adding the relations (4) and (5) and rearranging the terms we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$
$$+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0 \tag{6}$$

since the terms involving $\frac{\partial p}{\partial y}$ and $\frac{\partial q}{\partial x}$ cancel.

This is a linear equation of the first order to determine F. Its integrals are the integrals of

$$\frac{dp}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p}=\frac{dq}{\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}p}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{\partial f}{0} \tag{7}$$

Any of the integrals of (7) will satisfy (6). We should take the simplest relation involving at least one of p and q for $F=0$, so that the values of p and q to be derived may be calculated easily.

Ex 24. *Solve* : $px+qy=pq$. (*Luck.* 1956)

The auxiliary equations are

$$\frac{dp}{p}=\frac{dq}{q}=\frac{dz}{-p\,(x-q)-q\,(y-p)}=\frac{dx}{q-x}=\frac{dy}{p-y}$$

From first two terms we get $y=aq$.

Therefore from the given equation $aqx+qy=aq^2$

$$\text{or}\quad q=\frac{(y+ax)}{a} \text{ and } p=(y+ax).$$

Putting these values in $dz=p\,dx+q\,dy$, we get

$$a\,dz=(y+ax)\,a\,dx+(y+ax)\,dy=(y+ax)\,(dy+a\,dx)$$

$$\therefore\quad az=\tfrac{1}{2}(y+ax)^2+b.$$

Differentiating with respect to a and b, we get

$$z=x\,(y+ax), \text{ and } 0=1$$

Hence there is no Singular Integral.

Writing $b=\phi(a)$, we get

$$az=\tfrac{1}{2}(y+ax)^2+\phi\,(a).$$

Differentiating with respect to a

$$z=x\,(y+ax)+\phi'\,(a).$$

Eliminating a we get the General Integral.

Ex. 25. *Solve* : $(p^2+q^2)\,y=qz.$ (*Agra* 1951)

Charpit's auxiliary equations give

$$\frac{dp}{-pq}=\frac{dq}{(p^2+q^2)-q^2}$$

$$\therefore\quad p\,dp+q\,dq=0.$$

Integrating $p^2+q^2=a^2$. Hence with the help of the given equation, we get

$$a^2\,y=qz \quad\text{or}\quad q=\frac{a^2y}{z} \text{ and } p^2=a^2-\frac{a^4\,y^2}{z^2}.$$

$$\text{Now}\quad dz=p\,dx+q\,dy=\frac{a}{z}\sqrt{z^2-a^2\,y^2}\,dx+\frac{a^2y}{z}\,dy$$

$$\therefore\quad \frac{z\,dz-a^2\,y\,dy}{\sqrt{z^2-a^2\,y^2}}=a\,dx.$$

Integrating, we get

$$(z^2-a^2\,y^2)^{1/2}=ax+b.$$

The Complete Integral is

$$z^2=a^2\,y^2+(ax+b)^2 \tag{1}$$

For Singular Integral we have

$$0=2ay^2+2x\,(ax+b) \tag{2}$$

$$\text{and}\quad 0=2\,(ax+b) \tag{3}$$

Eliminating a and b between (1), (2) and (3) we get $z=0$, which satisfies the given equation.

Writing $b=\phi(a)$ in (1) and differentiating equation (1) with respect to a

$$0=2ay^2+2\{(ax+\phi(a)\}\{x+\phi'(a)\} \qquad (4)$$

Eliminating a between (1) and (4) we get the General Integral.

Ex. 26. *Find the Complete Integral of*

$$px+qy=z\ (1+pq)^{1/2}.$$

Charpit's equations give

$$\frac{dp}{p-p\ (1+pq)^{1/2}}=\frac{dq}{q-q\ (1+pq)^{1/2}} \quad \text{or} \quad \frac{dp}{p}=\frac{dq}{q}$$

$\therefore\ p=aq.$

Substituting in the given equation, we get

$$q\ (ax+y)=z\ (1+aq^2)^{1/2}$$

$$\text{or} \quad q^2\ [\ (ax+y)^2-z^2\ a\ ]=z^2$$

$$\therefore\ q=\frac{z}{[\ (ax+y)^2-az^2\ ]^{1/2}}$$

Hence we get

$$dz=p\ dx+q\ dy=(a\ dx+dy)\ q$$

$$\therefore\ \frac{dz}{z}=\frac{a\ dx+dy}{[\ ax+y)^2-az^2\ ]^{1/2}}=\frac{du}{[\ u^2-z^2\ ]^{1/2}},\ \text{where}\ \sqrt{a}\,.\,u=ax+y$$

$$\text{or} \quad \frac{du}{dz}=\frac{(u^2-z^2)^{1/2}}{z}$$

Putting $u=vz$, we have

$$z\ \frac{dv}{dz}=\sqrt{v^2-1}-v$$

$$\therefore\ \frac{dz}{z}=\frac{dv}{\sqrt{v^2-1}-v}=-[\ \sqrt{v^2-1}+v\ ]\ dv$$

Hence the Complete Integral is

$$\log z+\frac{v^2}{2}+\frac{v}{2}\ \sqrt{v^2-1}+\tfrac{1}{2}\log(v+\sqrt{v^2-1})=c,$$

where $v=u/z=(ax+y)/z\sqrt{a}$.

Ex. 27. *Solve* : $p=(qy+z)^2$

The auxiliary equations are

$$\frac{dp}{2p\ (qy+z)}=\frac{dq}{4q\ (qy+z)}=\frac{dy}{-2y\ (qy+z)}$$

$$\therefore \quad \frac{dp}{p}=\frac{dy}{-y} \text{ whence } py=a \text{ or } p=\frac{a}{y}$$

Putting this value of q in the given equation

$$q=\frac{1}{y}\left(\sqrt{\frac{a}{y}}-z\right)$$

$\therefore$ Writing the value of p and q in the expression $dz=pdx+qdy$

$$dz=\frac{a}{y}\,dx+\frac{dy}{y}\left[\sqrt{\frac{a}{y}}-z\right]$$

$$ydz+zdy=adx+dy\sqrt{\frac{a}{y}}$$

or $\quad yz=ax+2\sqrt{ay}+b$

Ex. 28. *Solve* : $(x^2-y^2)\,pq-xy\,(p^2-q^2)-1=0$

Charpit's auxiliary equations are

$$\frac{dp}{2xpq-y\,(p^2-q^2)}=\frac{dq}{-2ypq-x\,(p^2-q^2)}=\frac{dx}{-(x^2-y^2)\,q+2pxy}$$

$$=\frac{dy}{-(x^2-y^2)\,p-2qxy}$$

using x, y, p and q as multipliers we get

$$xdp+ydq+pdx+qdy=0$$

or $\quad (xdp+pdx)+(ydq+qdy)=0$

i.e. $\quad xp+yq=a \qquad \ldots\ldots(1)$

$\therefore \quad p=\dfrac{a-qy}{x}$. Putting this value of p in the given equation

we get $\qquad (x^2-y^2)\,\dfrac{a-qy}{x}\,q-\dfrac{(a-qx)^2}{x}\,y=1$

giving $\qquad q=\dfrac{a^2y+x}{a(x^2+y^2)} \qquad \ldots\ldots(2)$

$\therefore$ From (1) $p=\dfrac{a-yq}{x}=\dfrac{a^2x-y}{a(x^2+y^2)}$

Then $\quad dz=pdx+qdy=\dfrac{a^2x-y}{a(x^2+y^2)}\,dx+\dfrac{a^2y+x}{a(x^2+y^2)}\,dy$

or $\qquad dz=a\,\dfrac{xdx+ydy}{x^2+y^2}+\dfrac{xdy-ydx}{a(x^2+y^2)}$

$\therefore \quad z=\frac{1}{2}\,a\log(x^2+y^2)+(1/a)\tan^{-1} y/x+b.$

Ex. 29. *Find the Complete Integral of*

$$pq+4\,(px+qy+z)=0. \qquad \textit{(Tripos)}$$

Auxiliary equations give $\dfrac{dp}{8p}=\dfrac{dq}{8q}$

$$\therefore \quad p=aq$$

With the help of the given equation, we get

$$aq^2+4\,(aqx+qy)+4z=0$$

or $\quad aq^2+4q\,(ax+y)+4z=0$

$$\therefore \quad q=\frac{-4(ax+y)+\sqrt{16(ax+y)^2-16\,az}}{2a},$$

taking one value

$$aq=-2\,(ax+y)+2\sqrt{(ax+y)^2-az}$$

$$ap=a^2\,q=-2a\,(ax+y)+2a\sqrt{(ax+y)^2-az}$$

$\therefore\ adz=apdx+aqdy$

$$=-2\,(ax+y)\,(adx+dy)+2\sqrt{(ax+y)^2-az}\,(adx+dy)$$

$\therefore$ Put $(ax+y)^2-az=t^2$ or $2\,(ax+y)\,(adx+dy)-adz=2t\,dt$

$$\therefore \quad 2\,adz+2\,tdt=t\,\frac{2\,tdt+adz}{\sqrt{t^2+az}}$$

Putting $az+t^2=u^2$, so that $adz+2\,tdt=2\,udu$

$$\text{we get } \frac{dt}{du}=\frac{2u-t}{t}$$

Now, let $t=vu$

$$\therefore \quad v+u\frac{dv}{du}=\frac{2-v}{v} \quad \text{or} \quad u\,\frac{dv}{du}=\frac{2-v-v^2}{v}$$

$$\therefore \quad -\frac{du}{u}=\frac{v}{v^2+v-2}\,dv=\frac{v\,dv}{(v+2)\,(v-1)}$$

$$\text{or} \qquad -\frac{3\,du}{u}=\left(\frac{2}{v+2}+\frac{1}{v-1}\right)dv$$

Integrating

$$\frac{A}{u^3}=(v+2)^2\,(v-1) \quad \text{or} \quad A=(uv+2u)^2\,(uv-u)$$

$\therefore\ A=(t+2u)^2\,(t-u)$, where

$$t^2=(ax+y)^2-az \text{ and } u^2=(ax+y)^2.$$

EXERCISE XI (I)

Solve the eqaations :

1. $q=xp+p^2$. (*Agra* '55, *Punjab* 56)
2. $q=2yp^2$.
3. $p^2-q^2=\dfrac{x-y}{z}$. (*Mysore* '48, *Agra* 1961)
4. $z=pq$. (*Agra* 1957, *Raj*. 1960)
5. $p^2+q^2-2px-2qy+1=0$.
6. $(p+q)(px+qy)=1$.
7. $x^2y^3z^3pq^2=1$. (*Agra* 1960)
8. $x^2y^3z^{-3}p^2q=1$.
9. $p^3-y^2q=y^2-x^2$.
10. $2xz-px^2-2qxy+pq=0$. (*Agra* 54, *Punjab* '54, *Nag*. '57)

Miscellaneous Exercise on Chapter XI

1. Solve the following linear equations :—

(*i*) $\cos(x+y)\,p+\sin(x+y)\,q=z$. (*Raj*. 1954, 1958)

(*ii*) $\dfrac{y^2z}{x}p+xzq=y^2$. (*Raj*. 1955)

(*iii*) $(mz-ny)\,p+(nx-lz)\,q=ly-mx$. (*Agra* 1949 '53)

(*iv*) $\dfrac{y-z}{yz}\,y+\dfrac{z-x}{xz}\,q=\dfrac{x-y}{xy}$ (*Agra* 1956, *Raj*. 1959)

(*v*) $x^2p+y^2q=z^2$. (*Raj*. 1951)

(*vi*) $x(y-z)\,p+y(z-x)\,q=z(x-y)$. (*Raj*. 1950)

(*vii*) $(x^3+3xy^2)\,p+(y^3+3x^2y)\,q=2(x^2+y^2)\,z$.

(*viii*) $x^2(y-z)\,p+y^2(z-x)\,q=z^2(x-y)$. (*Agra* '32)

(*ix*) $(x^2-yz)\,p+(y^2-xz)\,q=(z^2-xy)$. (*Nag*. '58, *Agra* '52)

(*x*) $(x^2-y^2-z^2)\,p+2xyq=2zx$.

(*xi*) $(y^2+z^2-x^2)\,p-2xy\,q+2xz=0$. (*Pun*. '57 *Agra* '57)

(*xii*) $(y^3x-2x^4)\,p+(2y^4-x^3y)\,q=9z(x^3-y^3)$. (*Agra* '51 '60, *Raj*. 60)

(*xiii*) $(x+2z)\,p+(4xz-y)\,q=2x^2+y$. (*Nagpur* '53)

2. Show that the complete integral of the differential equation $q-p+x-y=0$, represents a family of paraboloids of revolution.

3. Find three complete integrals of $pq=px+qy$.

4. Fing a complete integral of each of the following :—

(*i*) $q=e^{-p/a}$. (*ii*) $p(1+q)=qz$.

(*iii*) $q=(z+px)^2$. (*Punjab* 1956) (*iv*) $3p^2-2q^2=4pq$.

(*v*) $z^2(p^2+q^2)=x^2+y^2$.

(*vi*) $pxy+pq+qy=yz$. (*Punjab* 1957)
(*vii*) $pz=1+q^2$. (*Agra* 1958)
(*viii*) $p^3+q^3-3pq\,z=0$. (*Agra* 1935)
(*ix*) $z=px+qy+\sqrt{(\alpha p^2+\beta q^2+\gamma)}$. (*Raj*. 1951)
(*x*) $z^2\,(p^2\,z^2+q^2)=1$ (*Nagpur* '57)
~~(*xi*)~~ $\cancel{pq=x^m\,y^n\,z^l}$. (*Raj*. 1961, *Agra* 1957, *Lucknow* '56)
(*xii*) $p^2+q^2=x^2+xy+y^2$. (*Agra* 1932)
(*xiii*) $2\,(pq+py+qx)+x^2+y^2=0$. (*Luck*. '49, *Agra* '47)
(*xiv*) $p^2+q^2-2pq\tanh 2y=\operatorname{sech}^2 2y$.
(*xv*) $p^2+q^2-2px-2qy+2xy=0$. (*Delhi* 1959, *Agra* 1961)

5. Find Singular Integrals in Q. 4. (*viii*) and (*ix*).

6. Derive by Charpit's method the integrals of the differential equations of the following forms :

(*i*) $Pp+Qq=R$. (*Lagrange's linear equation*)
(*ii*) $F\,(p,q)=0$. (*iii*) $F\,(z,p,q)=0$.
(*iv*) $f\,(x,p)=F\,(y,q)$. (*v*) $z=px+qy+f\,(p,q)$.

ANSWERS

EXERCISE I (Page 5)

1. (c) $\dfrac{d^2y}{dx^2}+m^2y=0$ (b) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$

(c) $y=x\dfrac{dy}{dx}+\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^3$ (d) $y-2x\dfrac{dy}{dx}-y\left(\dfrac{dy}{dx}\right)^2=0.$

4. (a) $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0.$ (b) $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-xy=0.$

EXERCISE II (A) (Page 8)

1. $xy+\frac{1}{2}x^2=c.$
2. $\dfrac{x}{y}+\frac{1}{2}x^2=c.$
3. $\sin x=y\,(x+c).$
4. $x^2-y^2-2xy=c;$
5. $\frac{1}{2}(x^2+y^2)+\tan^{-1} y/x=c.$
6. $x^4+y^4+6x^2y^2=c.$
7. $y\,(1+\log x)=1+cxy.$
8. $y\,(1-x^2)=\frac{1}{2}x^2-\frac{1}{4}x^4+c.$
9. $y/x+\sin(1/x)=c.$
10. $(x^2+y^2+z^2)^{1/2}+\tan^{-1}\dfrac{x}{z}+ax^3+by^3+cz=k.$
11. $x^2+y^2+2\tan^{-1}\dfrac{y}{x}=c.$
12. $\log(x^2+y^2)^{1/2}-m\tan^{-1}\dfrac{y}{x}=c.$

EXERCISE II (B) (Page 9)

1. $x\tan x-\log\sec x=y\tan y-\log\sec y+c.$
2. $3\,(e^y-e^x)=x^3+c.$
3. $y=1+c\,e^{1/x}.$
4. $y\tan^{-1}x=c.$
5. $(x^2+1)\,(y^2+1)=c.$
6. $e^x+e^{-y}+\frac{1}{3}x^3=c.$
7. $2\tan^{-1}(1+\tan\frac{1}{2}x)=c+\log(1+2\tan\frac{1}{2}y)$
8. $x^2+y^2+x\sin 2x+y\sin 2y+\frac{1}{2}\cos 2x+\frac{1}{2}\cos 2y=c.$
9. $(1+y)\,(1+e^x)=ce^y$
10. $(1+\log y)/y+x^2\cos x-2x\sin x-2\cos x=c.$
11. $y^2\log y=x\sin x+c.$
12. $[\log(\sec x+\tan x)]^2-[\log(\sec y+\tan y)]^2=c.$

13. $\log(x/y)-(y+x)/xy=c.$

14. $y\sin y=x^2\log x+c.$

15. $\tan y=c\,(1-e^x)^3.$

16. $y=c\,(1-ay)\,(x+a)$

EXERCISE II (C) (Pages 12-13)

1. $x-y+\log(x+y)=c.$

2. $6y-3x=\log(3x+3y+2)+c.$

3. $x+y-4\log(2x+3y+7)=c.$

4. $$\frac{1}{\sqrt{3}}\tan^{-1}[(x+2y)/x\sqrt{3}]=c+\tfrac{2}{3}\log(x-y)+\tfrac{1}{6}\log(x^2+xy+y^2)$$

5. $xy\cos(y/x)=c.$

6. $x^2+y^2=cx.$

7. $x=c\sin y/x$

8. $2y-x+c=\log(x-y+2)$

9. $6y-3x=\log(3x+3y+2)+c.$

10. $y/x=c+\log y.$

11. $\log x=\cos(y/x)+c$

12. $\log y=c+x^2/2y^2.$

13. $x^2y=c^2\,(y+2x)$

14. $y-x=c\sqrt{xy}$

15. $(5y-2x-3)^4=c\,(4y-4x-3)$

16. $(3y-5x+10)^2=c\,(y-x+1)$

17. $4\,(x-2y)=\log(4x+8y+5)+c.$

18. $cx^2=y+\sqrt{x^2+y^2}$

19. $x^4+y^4+6x^2y^2=c.$

20. $x^2-y^2=cx.$

21. $xy+y^2-3y=x^2+x+c.$

22. $$\tan^{-1}\left\{\frac{2y+1}{2x+1}\right\}=\log\left\{c\sqrt{x^2+y^2+x+y+\tfrac{1}{2}}\right\}$$

23. $\log(x-y-1)=x-2y+c.$

EXERCISE II (D) (Pages 16-17)

1. $y=\sin x+c/\sin x.$

2. $y=\tan x-1+c\,e^{-\tan x}.$

3. $yx\sec x=\tan x+c.$

4. $2xy+\cos x^2=c.$

5. $y\log x=c+(\log x)^2.$

6. $y\cot x=c+\log\tan\tfrac{1}{2}x.$

7. $xy+\tan^{-1}y=c.$

8. $x-1=1/y+c\,e^{1/y}.$

9. $12y=e^{x^3}(2x^3-1)+c\,e^{-x^3}.$

10. $\sin y=(1+x)\,(e^x+c).$

11. $2x=e^y\,(1+cx^2).$

12. $x^2y=c+(2-x^2)\cos x+2x\sin x.$

13. $x=c\,e^{-\tan^{-1}y}+\tan^{-1}y-1.$

14. $xy=c-\tan^{-1}x.$

15. $y\sqrt{1+x^2}=c+\tfrac{1}{2}\log\tan(\tfrac{1}{2}\tan^{-1}x).$

Another form of solution is

$$y\sqrt{1+x^2}=c+\tfrac{1}{2}\log\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}.$$

16. $\dfrac{1}{x \sin y}=\dfrac{1}{2x^2}+c.$ 17. $\dfrac{1}{x \log y}=\dfrac{1}{2x^2}+c.$

18. $e^{-(n-1)y}=1+c\ e^{(n-1)x^2/2}$ 19. $y^{-1} e^{x}=c-x^2.$

20. $y^{-2}=-1+(c+x) \cot \left(\dfrac{\pi}{4}+\dfrac{x}{2}\right);$

21. $e^{-n+1}=c\ e^{(n-1)\sin x}+2 \sin x+\dfrac{2}{n-1};$

EXERCISE II (E) (Page 19)

1 $x^2+y^2-2a^2 \tan^{-1} \dfrac{y}{x}=c.$ 2. $x+2x^2y+2xy^2+y=c.$

EXERCISE II (F) (Pages 24-25)

1. $x/y+\log (y^3/x^2)=c.$
2. (a) $\frac{1}{2} x^2y^2+\log (x/y)-1/xy=c.$
 (b) $2 \log x-\log y=c+1/xy.$
3. $xy+y^2+2x/y^2=c.$
4. (a) $x^2-y^2=cx.$ (b) $(4x^5+2x^4y+\frac{4}{3} x^3 y^2+x^3 y^3) y=c.$
5. (a) $4 (xy)^{1/2}-\frac{2}{3} (y/x)^{3/2}=c.$ (b) $x^2 y^3 (1+2xy)=c.$

Miscellaneous Examples on Chapter II (Pages 25-26)

1. $(x^2+y^2)^{1/2}=a \sin [c+\tan^{-1} (y/x)]$
2. $(b-a) \log [(x+y)^2-ab]=2 (x-y)+c.$
3. $a \log [(x-y-a)/(x-y+a)]=2y+c.$
4. $y=a \tan^{-1} [(x+y)/a]+c.$
5. $4x+y+1=2 \tan (2x+c).$
6. $x \log y=e^x (x-1)+c.$ 7. $y=x \sinh (x+c).$
8. $xe^{\tan^{-1} y}=\tan^{-1} y+c.$ 9. $x^2=2cy+c^2.$
10. $x^4-y^4+2x^2 y^2-2a^2 x^2-2b^2 y^2=c.$ 11. $x^2-y^2-1=cx.$
12. $\log (x^2+y^2)^{1/2}=m \tan^{-1} (y/x)+c.$
13. $y^2/x=ay^{n+2}/(n+2)+c.$ 14. $\frac{2}{3} x^3y-\frac{1}{2} y^3+e^{x}=cy.$
15. $xg=k^2 \log \cosh (gt/k).$
16. $y^2 e^{2bx}=\dfrac{2a\, e^{2bx}}{(4b^2+1)^{1/2}} \cos [x-\tan^{-1} (1/2b)]+c.$
17. $\frac{1}{2} e^{2y}=\frac{1}{3} e^{3x}+\frac{1}{2} x^3+c.$ 18. $x-2y+\frac{1}{2} \log (x^2+y^2)=c.$
19. $x^2 y^3 (1+2xy)=c.$ 20. $y (x+\log x)+x \cos y=c.$
21. $x+y+\frac{4}{3}=c\ e^{3(x-2y)}.$ 22. $y \sin y=x^2 \log x+c.$
23. $1-e^{-3s}=c\ e^{x^3}$
24. Put $e^x=u$, $e^y=v$. *Ans.* $e^y=c\ e^{-e^x}+e^x-1.$

25. $x+c=\log [1+\tan \frac{1}{2}(x+y)]$
26. $2x=\sin y\,(1-2c\,x^2)$
27. $x^3+y^3=3\,(axy+c)$.
28. $x^2y+e^x=cy$.
29. $cy=e^{y/x}$

30. $y=x/(1-x^2)^{1/2}+c\,e^{-x/(1-x^2)^{1/2}}$
31. $x^2-y-1-x\cos y=cx$.

32. $\tan y=ce^{-x^2}+\frac{1}{2}(x^2-1)$.
33. $ax^2+fy^2+2bxy+2ey+2cx=\text{constant}$.

EXERCISE III (A) (Page 31)

1. $y=c_1\,e^{nx}+c_2\,e^{-nx}$.
2. $y=c_1\,e^{x}+c_2\,e^{2x}+c_3\,e^{-x}$.
3. $x=c_1\,e^{(3/2)t}+c_2\,e^{-4t}$
4. $y=c_1\,e^{6x}+c_2\,e^{-9x}$
5. $x=c_1\,e^{2y/3}+c_2\,e^{-8y/3}$.

EXERCISE III (B) (Page 32)

1. $y=(c_1+c_2\,x)\,e^{x}+c_3\,e^{-3x}$
2. $y=(c_1+c_2\,x)\,e^{-x}$
3. $y=(c_1+c_2\,x+c_3\,x^2)\,e^{x}$.
4. $y=(c_1+c_2\,x)\,e^{x}+(c_3+c_4\,x)\,e^{-x}$.

EXERCISE III (C) (Page 33)

1. $y=(c_1+c_2\,x)\cos 2x+(c_3+c_4\,x)\sin 2x$.
2. $y=c_1\,e^{x}+(c_2\cos x+c_3\sin x)\,e^{-2x}$
3. $y=c_1\,e^{-x/2}+(c_2\cos x+c_3\sin x)\,e^{x}$.
4. $y=c_1\,e^{x}+c_2\,e^{-x}+c_3\cos x+c_4\sin x$.
5. $y=(c_1+c_2\,x)\cos x+(c_3+c_4\,x)\sin x$.

EXERCISE III (D) (Page 37)

1. $y=c_1\,e^{3x}+c_2\,e^{2x}+\frac{1}{2}\,e^{4x}$.
2. $y=c_1\,e^{x}+c_2\,e^{-x}-2-5x$.
3. $y=c_1\,e^{3x}+c_2\,e^{-5x}-x^2-\dfrac{4}{15}\,x-\dfrac{38}{225}$.

EXERCISE III (E) (Page 39)

1. $y=e^{x}\,(c_1\,x+c_2)+\frac{4}{3}\,e^{5x/2}$.
2. $y=c_1\,e^{-x/2}\cos(\frac{1}{2}\,x\sqrt{3}+c_2)+e^{-x}$.
3. $y=e^{-px}\,(c_1\cos qx+c_2\sin qx)+e^{kx}/\{(p+k)^2+q^2\}$

EXERCISE III (F) (Pages 42-43)

1. $y=c_1\sin(3x+c_2)+\frac{1}{5}(\cos 2x+\sin 2x)$.
2. $y=c_1\cos(ax+\alpha)+\dfrac{x}{2a}\sin ax+\dfrac{\cos bx}{a^2-b^2}$
3. $y=c_1\cos(2x+c_2)-\frac{1}{4}\,x\cos 2x+\frac{1}{5}\,e^{x}$.

4. $y=c_1 e^{-3x}+c_2 e^{2x}+c_3 e^{-2x}-\frac{\sin 4x}{125}-\frac{3\cos 4x}{500}$

5. $y=c_1 e^{2x}+c_2 e^{-2x}-\frac{8}{17}\sin\frac{1}{2}x.$

6. $y=c\,e^{-x}+e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$

$$+\frac{\sin 3x+27\cos 3x}{730}-\frac{1}{2}+\frac{\sin x-\cos x}{4}$$

EXERCISE III (G) (Page 45)

1. $y=c_1+c_2 e^{2x}\cos(x+c_3)+\frac{2}{5}x.$
2. $y=(c_1+c_2 x)e^{x}+c_3 e^{-x}+x+1.$
3. $y=c_1+c_2 e^{-x}+c_3 e^{3x}-\frac{1}{18}x^3+\frac{1}{36}x^2-\frac{25}{108}x.$
4. $y=c_1+(c_2+c_3 x)e^{-x}+\frac{1}{18}e^{2x}+\frac{1}{6}x(2x^2-9x+24).$

EXERCISE III (H) (Page 47)

1. $y=(c_1+c_2 x+c_3 x^2)e^{x}+c_4 e^{-2x}+\frac{1}{18}x^3 e^{x}.$
2. $y=e^{x}(c_1\cos 2x+c_2\sin 2x)-\frac{1}{10}e^{2x}(\cos x-2\sin x).$
3. $y=c_1 e^{x}(\cos\sqrt{3}x+c_2)+\frac{1}{2}e^{x}\cos x.$
4. $y=c_1 e^{x}+c_2 e^{-x}+\frac{1}{5}(2\sinh x-\sin x-\cosh x\cos x).$
5. $y=c_1 e^{-x}+c_2 e^{-2x}+c_3 e^{3x}-\frac{1}{12}e^{2x}\left(x+\frac{17}{12}\right).$
6. $y=c_1 e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x+c_2)+c_3 e^{x/2}(\cos\frac{1}{2}\sqrt{3}x+c_4)$

$$+ax^2-2a-\frac{1}{481}b\,e^{-x}(9\sin 2x+20\cos 2x).$$

7. $y=e^{2x}\left(c_1+\frac{1}{16}x^2-\frac{9}{64}x\right)+c_2 e^{-6x}.$

Miscellaneous Examples on Chapter III (Page 50)

1. $y=(c_1+c_2 x)e^{-x}+\frac{1}{2}(x-1)\sin x+\frac{1}{2}\cos x.$

2. $y=(c_1+c_2 x)\sin x+(c_3+c_4 x)\cos x+\frac{1}{12}x^3\sin x$

$$+\frac{1}{48}(9x^2-x^4)\cos x$$

3. $y=c_1 e^{-x}+c_2 e^{x}+c_3\cos(x+c_4)+\frac{1}{8}(x^2\cos x-3x\sin x)$.
4. $y=(c_1+c_2 x)e^{x}-e^{x}(2\cos x+x\sin x)$.
5. $y=e^{2x}(c_1+c_2 x+3\sin 2x-2x^2\sin 2x-4x\cos 2x)$.
6. $y=c_1\cos x+c_2\sin x+\frac{1}{2}e^{-x}+\frac{1}{2}x\sin x$.
$+x^3-6x-\frac{1}{5}e^{x}(2\cos x-\sin x)$.
7. $y=e^{-x/2}\left[\{\cos(x\sqrt{3/2})\}(\frac{1}{4}x+c_1)\right.$

$$+\{\sin(x\sqrt{3/2})\}\left(c_2+\frac{1}{12}\sqrt{3}\,x\right)$$

$$+c_3 e^{x/2}\cos(x\sqrt{3/2}+c_4).$$

8. $y=(c_1+c_2 x+\frac{1}{4}x^2)e^{x}+(c_3+c_4 x)\sin x$.

$$+(c_5+c_6 x)\cos x-\frac{1}{32}x^2\sin x+\frac{1}{2}.$$

9. $y=e^{-\sqrt{7}x/2}\cos\left(\frac{3}{2}x+c_2\right)$

$$+c_3 e^{\sqrt{7}x/2}\cos\left(\frac{3}{2}x+c_4\right)+x^2+\frac{127}{8}$$

10. $y=A\cos(x-a)+\frac{3}{2}-\frac{1}{2}\cos 2x-\frac{3}{4}x\cos x+\frac{1}{16}\sin 3x$.
11. $y=A\cos(x-a)+B\cos(3x-\beta)-3x\cos x+x\cos 3x$.
12. $y=c_1 e^{-x}\cos(3x+c_2)+6\cos 3x-\sin 3x\ .\ y=1$.
13. $y=-x^3\cos x+3x^2\sin x$.
14. $y=e^{-x}(c_1\cos x+c_2\sin x)+c_3 e^{3x}\cos(2x+c_4)+c_5 e^{-4x}$.

EXERCISE IV (A) (Page 53)

1. $(y-4x-c)(y-3x-c)=0$. 2. $(y-2x-c)(y-3x-c)=0$.
2. $(y-6x+c)(y-3x+c)=0$. 4. $(2y-x^2-c)(2y+3x^2-c)=0$
5. $y(1\pm\cos x)=c$. 6. $(y-e^{x}-c)(y+e^{-x}-c)=0$
7. $(2y+x^2-c)(y+x+1-c\,e^{x})=0$.
8. $(y-x+c)(x^2+y^2-c^2)=0$. 9. $(2y-x^2-c)(2x-y^2-c)=0$.
10. $(y-x+c)(xy+c)=0$. 11. $343(y+c)^3=27ax^7$.
12. $(2y+x^2-c)(x+\log y-c)=0$.
13. $(2y-x^2-c)(y-c\,c^{x})(y+x-1-c\,e^{-x})=0$.
14. $(y+x-1+c\,e^{-x})(2xy+x^2+c)(y+x^2+c)=0$.
15. $(y-c)(xy+cy-1)(y-c\,e^{1/x})=0$.
16. $(y-cx^2)(yx^3-c)=0$.

17. $(y-c)(y+x^2-c)(1+xy+cy)=0$.
18. $y^2(1-y)=(x+c)^2$.

EXERCISE IV (B) (Page 55)

1. $y=3x-a\log(\frac{1}{3}-c\,e^{3x/a})$.
2. $x=p(c+\cosh^{-1}p)(p^2-1)^{-1/2}$; with the given relation.
3. $x+c=\frac{1}{2}a[\log(1+p^2)^{-1/2}(p-1)-\tan^{-1}p]$; with the given relation.
4. $x=4p^3+\frac{1}{2}p^{-2}+c$; $y=3p^4+p^{-1}$.
5. $(p-1)^2x=c-p+\log p$; $(p-1)^2y=p^2(c-p+\log p)+p$.
6. $2y=cx^2+\text{const.}$
7. $y=\log(p^3+p)$; $x=2\tan^{-1}p-p^{-1}+c$
8. $\cos[\{\sqrt{(1-c^2+2cx-x^2)}-y\}/(c-x)]=c-x$.
9. $y=(x+c)\tan\frac{1}{2}x$.
10. $x=\tan p+c$; with the given relation.

EXERCISE IV (C) (Page 56)

1. $y=p+(c+\cosh^{-1}p)(p^2-1)^{-1/2}$, and the given relation.
2. $(2x-b)c=y^2-ac^2$.
3. $y=c-a\log(p-1)$; $x=c+a[\log p-\log(p-1)]$.
4. $y^2=2cx+c^2$.
5. $y-c=\sqrt{x-x^2}-\tan^{-1}\sqrt{\dfrac{1-x}{x}}$
6. $2y+c=a[p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})$, $x=a\sqrt{1+p^2}$.
7. $\log y=cx+c^2$.

EXERCISE IV (D) Page 57)

1. $y=cx+a/c$.
2. $y=cx+c-c^2$
3. $y=cx+ac(1-c)$.
4. $y=cx+(1+c^2)^{1/2}$.
5. $y=cx+\sqrt{(b^2-a^2c^2)}$.
6. $(y-cx)(c-1)=c$.
7. $xc^2-yc+a=0$.
8. $y=c(x-b)+a/c$.
9. $y=xc+c^3$.
10. $y^2=cx+c^2$.

EXERCISE IV (E) (Pages 63-64)

1. $y^2=2cx+c^2$.
2. $(y+c)^2+(x-a)^2=1$.
3. $y=x\sin(\log cx)$.
4. $x^2+y^2=cx$.
5. $y(1+p^2)^{3/2}=c$; the given relation
6. $(x+c)(p^2+1)^2=a(p^2-1)$, the given relation.
7. $\tan^{-1}y/x+c=\text{vers}^{-1}2a\sqrt{x^2+y^2}$.
8. $y=4c(cxy+1)$.
9. $(2y-x^2-c)(y+\sin x-c)(y+2\cos x+k)=0$.
10. $\sin^{-1}y/x+\sin^{-1}1/x=c$ or $(y-cx^2)^2=(x^2-1)(x^2-y^2)$.
11. $\frac{1}{2}\log(x^2+y^2)+\tan^{-1}y/x=c$.
12. $c^2+2cx(3a^2y^2-8x^2)-3a^4x^2y^4+a^6y^6=0$.
13. $y^2=cx^2+c^2$.

14. $(y-\sin^{-1} x/a-c)\ (y-\cos^{-1} x/a-c)\ (\sqrt{y}-\sqrt{x}-c)=0.$
15. $c^2\ (x^2-a^2)-2cxy+y^2+a^4=0.$
16. $x\ (1+p^2)^{1/2}=p\ [\ c+a \log (p+\sqrt{1+p^2})\ ].$
17. $(y-cx^2)\ (y^2+3x^2-c)=0.$
18. $a^2\ c^2-12acxy+8cy^3-12x^2\ y^2+16ax^3=0.$
19. $x=a \log [\ ay+\sqrt{(a^2+y^2-1)}\ ]+\log [\ y-\sqrt{(a^2+y^2-1)}\ ]+c.$
20. $y^2=Ax^2+\dfrac{cA}{b+aA}.$
21. $x=a \log [\ \sqrt{(a^2+4by)}-a\ ]+\sqrt{(a^2+4by)}+c.$
22. $(y-c\ e^x)\ (e^y+e^x-c)\ (y-x^2-c)=0.$
23. $y+2y^3-\frac{3}{2}\ x^2\ y^2+\frac{1}{3}\ x^3=c.$
24. $y=cx \pm \sqrt{(1-c^2)}.$
25. $xy-\dfrac{1}{xy}-2 \log y=c.$
26. $y=2c\sqrt{x}+f\ (c^2).$
27. $y^2-c+\dfrac{c}{f(c)}\ x^2=0.$
28. $y=cx^2+x\,f\,(c).$
29. $xy \cos (y/x)=c.$
30. $y=cx^2-\dfrac{c}{c+1}\ h^2.$

EXERCISE V (Pages 76-77)

1. $x^2\ (y^2-4x^3)=0.$
2. $y^2=ax^2.$
3. $y^2+m^2\ x^2=m^2.$
4. $x^2=4y.$
5. No singular solution, $x=0$ is the cuspidal locus.
6. $x=0$, $x=1$, $x=2$ are singular solutions
   $x=1 \pm 1/\sqrt{3}$ are tac-loci.
7. $4y^3+27\ x^3=0$, also $x=0$ is partly singular solution and partly cuspidal locus.
8. $6y=x^3$; $x=0$ is cuspidal locus.
9. $y^2=1.$
10. $y=1$ is the singular solution, $y=0$ is nodal locus and $y=\frac{8}{9}$ is tac-locus.
11. $x=0$ is sing. sol., $x=\frac{1}{3}$ tac-locus, $x=1$ nodal locus.
12. $x=0$ is sing. sol., $x=a$ tac-locus, $x+3a=0$ nodal locus.
13. $y^2=x^2.$
14. $(3y+x)\ (y-x)=0.$
15. $x^4+4y=0$ is sing. sol, $x=0$ is tac-locus.
16. $y=0,\ y+4x^2=0.$
17. $x^2+y^2=a^2$ is sing. sol, $x=0$ tac-locus.
18. $x^4-16y=0,\ y=0.$
19. No sing. sol., $y=\pm 1$ cuspidal locus.
20. $y+x^2=0$ is cuspidal locus, no singular solution.
21. $4y^2\ x^3=27.$
22. $e^{2x}+e^{2y}=1,$
23. $4ax^2\ y^2+(x^2-ay^2-b)^2=0.$
24. $4y+1=0.$
25. $y=cx \pm \sqrt{(a^2\ c^2+b^2)}$; $b^2\ x^2+a^2\ y^2=a^2\ b^2$ is sing. solution.
26. $y^2=cx^2+c/(c-1).$
27. Put $x^2=X,\ y^2=Y.$
31. $ab\ (xy-1)-bx+ay=0.$

EXERCISE VI (A) (Page 81)

1. $y=c_1 x+c_2+x^{-2}$. 2. $y=x (c_1+c_2 \log x)+2 \log x+4$.

3. $y=c_1 x^{-1}+c_2 x^{\sqrt{2}}+c_3 x^{-\sqrt{2}}$

4. $y=c_1 x^3+c_2+c_3 \log x$.

5. $y=c_1 x+c_2 x^2+c_3 x^3-\frac{85}{108}-\frac{1}{6} (\log x)^2-\frac{11}{18} \log x$.

6. $y=c_1 x^2+x^{5/2}\left(c_2 x^{\sqrt{21}/2}+c_3 x^{-\sqrt{21}/2} \right)-\frac{1}{5} x^3$.

EXERCISE VI (B) (Page 83)

1. $y=c_1 x^3+c_2 x^{-1}+c_3$

2. $y=[c_1+c_2 \log x+c_3 (\log x)^2] x^2$

3. $y=x [c_1 \cos (2 \log x)+c_2 \sin (2 \log x)]$.

4. $y=c_1 \cos (\log x)+c_2 \sin (\log x)+c_3$.

EXERCISE VI (C) (Pages 93–94)

1. $y=\sqrt{x} [c_1 \cos (\frac{1}{2}\sqrt{3} \log x)+c_2 \sin (\frac{1}{2}\sqrt{3} \log x)]+x^2$.

2. $y=c_1 x^{-5}+c_2 x^{-1}+\frac{1}{60} x^5$.

3. $y=(c_1+c_2 \log x) x^{-2}+\frac{1}{36} x^4$. 4. $c_1 x^2+c_2 x^3+\frac{1}{2} x^4$.

5. $y=c_1 x^4+c_2 x^{-1}+\frac{1}{5} x^4 \log x$. 6. $y=c_1 x+c_2 x^{-1}+x^m/(m^2-1)$

7. $y=x^2 (c_1+c_2 \log x)+x^m/(m-2)^2$

8. $y=c_1+c_2 x^{-1}-\log x+\frac{1}{2} (\log x)^2$

9. $y=c_1 x^{-1}+c_2 x^{-2}+x^{-2} e^x$.

10. $y=c_1 x+c_2 x^{-3/2}+\frac{1}{5} x \int (X/x^2) dx-\frac{1}{5} x^{-3/2} \int x^{1/2} X dx$.

11. $y=c_1 \cos (\log x)+c_2 \sin (\log x)+c_3+\frac{1}{10} x^2$.

12. $y=(c_1+c_2 \log x) \cos (\log x)+(c_3+c_4 \log x) \sin (\log x)+x$.

13. $y=x^{-1} [c_1 \cos (\log x)+c_2 \sin (\log x)+\frac{1}{30} x^2+\frac{3}{10} x-2+c_3 x^{-1}$

14. $y=c_1 x^{-5}+c_2 x^4-\left(\frac{1}{14}x^2+\frac{1}{9} x+\frac{1}{20}\right)$.

15. $y=x^2 [c_1+c_2 \log x+c_3 (\log x)^2]+\frac{1}{6} x^2 (\log x)^3-1/(64 x^2)$.

16. $y=x [c_1+c_2 \log x+c_3 (\log x)^2+c_4 (\log x)^3]$
$+x (\log x)^4/4 !+\log x-4$.

17. $y=x [c_1 \cos (\log x)+c_2 \sin (\log x)]+x \log x$.

18. $y=x^2 [c_1 \cos (\log x)+c_2 \sin (\log x)]-\frac{1}{2} x^2 [\log x \cos (\log x)]$

19. $y=c_1 x^{-1}+x [c_2+c_3 \log x-(\log x)^2/8+(\log x)^3/12]$

20. $y=(c_1+c_2 \log x) \cos (\log x)+(c_3+c_4 \log x) \sin (\log x)$
$+(\log x)^2+2 \log x-3$.

21. $y=(5+2x)^2\,[\,c_1\ (5+2x)^{\sqrt{2}}+c_2\ (5+2x)^{-\sqrt{2}}\,]$.
22. $y=c_1+c_2 \log(x+1)+[\log(x+1)]^2+x^2+8x$.

EXERCISE VII (A) (Pages 104-105)

1. $y=x\,e^{-2x}\,(\int c_1\,x^{-2}\,e^{2x}\,dx+c_2)$.
2. $e^{e^x}\ (y-1)=c_2 \int e^{e^x}\ dx+c_2$.
3. $y\sqrt{(1+x^2)}=c_1 \log(x+\sqrt{1+x^2})+c_2$.
4. $y=x^{-1}\,[\,c_1 \cos(\log x)+c_2 \sin(\log x)\,]$
$+1/30\,x^2+3/10\,x-2+c_1/x)$.
5. $e^{\frac{1}{2}x^2}=c_1\,x^5 \int e^{\frac{1}{2}x^2}\,x^{-5}\,dx+c_2\,x^5 \int e^{\frac{1}{2}x^2}\,x^{-6}\,dx+c_3\,x^5$.
6. $e^{2e^x}\ y=\frac{1}{3} \int e^{2e}\ x^3\,dx+c_1 \int e^{2e^x}\,dx+c_2$.
7. $y\,(x-1)^5=c_1\,(x^4-6x^2+2x-\frac{1}{3})+x^3\,(c_2-4c_1 \log x)$.
8. $y=c_1\,(4x^3-2x^2-\frac{2}{3}\,x-\frac{1}{3})+x^3\,(x-1)\,[\,c_2-4c_1 \log x/(x-1)\,]$.
9. $y=c_1\,x+\sqrt{(1-x^2)}\,(c_2-\sin^{-1} x)$ or
$y=c_1\,x+\sqrt{(x^2-1)}\,[\,c_2-\log(x+\sqrt{x^2-1})]$.
10. $y\,(2x+3)=c_1 \log x+c_2+e^x$. 11. $y^2=c_1 \log x+c_2$.
12. $yx=(1/6a)\,x^3+c_1\,(a-bx)^3+c_2$.
13. $y=c_1 \sin^2 x+c_2 \cos x-c_2 \sin^2 x \log \tan \frac{1}{2}x$.
14. $x^3\,y_2+x^2\,y_1+x^3\,y=x^2+c_1$.
15. $x^5\,y_5-4x^4\,y_4+16\,x^3\,y_3-48x^2\,y_2+96xy_1$
$-96y+xy=x \log x-x+c_1$.
16. $x^3\,y_2+x^2\,y_1+x^3\,y=x^2+c$.
17. $x^3\,y_1+3x\,(1-x)\,y=\frac{1}{3}\,x^3+2 \log x+5/x^2$.
18. $y_2=\int f(x)\,dx+c_1,\ xy_2-y_1=\int x f(x)\,dx+c_2,$
$x^2\,y_2-2xy_1+2y=\int x^2 f(x)\,dx+c_3$.
19. $x^2 \dfrac{d^2y}{dx^2}-x \dfrac{dy}{dx}+xy^2=c$.

EXERCISE VII (B) (Page 106)

1. $y=\frac{1}{6}\,x^3-\sin x+c_1\,x+c_2$. 2. $y=(x-2)\,e^x+c_1\,x+c_2$.
3. $y=\log \sec x+c_1\,x+c_2$. 4. $y=\frac{1}{2} \log x+c_1\,x^2+c_2\,x+c_3$.
5. $y=ax \log x+c_1\,x+c_2$.
6. $y=c_1\,x^2+c_2\,x+c_3+1/12\,x^3+1/16 \sin 2x$.
7. $y=\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a^2 \sinh^{-1} x/a+c_1\,x+c_2$
8. $y=-\log x-\frac{1}{2}(\log x)^2+c_1\,x+c_2$.

EXERCISE VII (B) (Pages 107-108)

1. $y=c_1 \sinh(x+c_2)$ 2. $(c_1\,x+c_2)^2+a=c_1\,y^2$.

3. $ax=\log(y+\sqrt{y^2+c_1})+c_2$ or $y=c_1 e^{ax}+c_2 \cdot e^{-ax}$.

4. $\sqrt{2}ax+c_2=\int \dfrac{dy}{\sqrt{c_1-\log y}}$.

5. $y=\pm\coth \dfrac{x-c}{\sqrt{2}}$. 6. $y=\log \sec x$.

EXERCISE VII (D) (Pages 109–110)

1. $y=c_1 \int e^{\frac{1}{2}x^2}\, dx+c_2$ 2. $y=c_1+\cosh(x+c_2)$.
3. $y=c_1 e^{-x}+c_2+\frac{1}{2}e^x$. 4. $y=c_1 \log x+c_2$.
5. $y=c_1 x^3+c_2 x^4+\frac{2}{3}x+c_3$. 6. $2ay+x^2=c_1\sqrt{a^2-x^2}+c_2$.
7. $y=c_2-ax+c_1\log[x+\sqrt{(1+x^2)}]$.
8. $y=ax+c_1[\sin^{-1}x+x\sqrt{(1-x^2)}]+c_2$.
9. $y=c_1+\log(x^2+c_2)$.
10. $y=c_2 e^x(x-1)+c_1 x+c_1\int x e^x \int x^{-1} e^{-x}(dx)^2+c_3$.
11. $y=\frac{1}{2}c x^2+x f(c)+c_1$.
12. $y=c_1\log x+c_2+x^{n+1}/(n+1)^2$
13. $y=c_1\sqrt{(a^2-x^2)}+x^2/(2a)+c_2$.

EXERCISE VII (E) (Page 112)

1. $\dfrac{c_1+y}{c_1-y}=e^{c_1(x+c_2)}$ 2. $y^2=x^2+c_1 x+c_2$.

3. $y=c_1 e^{x/c_2}-c_2$. 4. $\sin(c_1-2\sqrt{2}y)=c_2 e^{-2x}$.
5. $e^{-ay}=c_1 x+c_2$. 6. $y^2+x^2+c_1 x+c_2=0$.
7. $cy+\sqrt{1+a^2c^2}=c_1 e^{cx}$.

EXERCISE VII (F) (Pages 113–114)

1. $a\log(y+c_2)=x+c_1$.
2. $ay=\frac{1}{2}[x\sqrt{x^2+c^2}+c^2\log(x+\sqrt{x^2+c^2})+c_2]$.
3. $15y=8(x+c_1)^{5/2}+c_2 x+c_3$.
4. $c e^{-y}=\cos(x+c_1)$ 5. $2c_1 y/a=c_1^2 e^{x/a}+e^{-x/a}+c_2$.
6. $y=b+(1/k^2)\log[\sec ak(x-c)]$.
7. $y+c_2=a\cosh(x+c_1)/a$. 8. $y=\log\sin(x-c_1)+c_2$.
9. Value of p is given by the relation
$$\log cx=\tfrac{1}{2}p^2+\tfrac{1}{2}p\sqrt{1+p^2}+\tfrac{1}{2}\sinh^{-1}p.$$

10. $y=\dfrac{2x^{n+1}}{(n+1)!}+2c_1\dfrac{x^n}{n!}+c_1^2\dfrac{x^{n-1}}{(n-1)!}+c_2x^{n-2}+c_3x^{n-3}+\dots+c_{n-1}x+c_n$

EXERCISE VII (G) (Page 115)

1. $y=c_1 e^{x/a}+c_2 e^{-x/a}+c_3 x+c_4$.
2. $y=c_1\sin(ax+c_2)+c_3 x+c_4$.

3. $y=c_1 e^{nx}+c_2 e^{-nx}+c_3+c_4 x+c_5 x^2+\dfrac{e^{ax}}{a^3(a^2-n^2)}$.

4. $y=c_1+c_2 x+x^{5/2} (c_3 x^{\frac{1}{2}\sqrt{1-4a^2}}+c_4 x^{-\frac{1}{2}\sqrt{1-4a^2}})$. where $a<\frac{1}{4}$;

$y=c_1+c_2 x+c_3 x^{5/2} \cos (\frac{1}{2}\sqrt{4a^2-1} \log x/c_4)$, where $a>\frac{1}{2}$.

5. $y=c_1+c_2 x+x^{5/2} (c_3 x^{\frac{1}{2}\sqrt{4\lambda+1}}+c_4 x^{-\frac{1}{2}\sqrt{1+4\lambda}})$.

EXERCISE VII (H) Page 118)

1. $y=nx \log (c_1+c_2/x)$.
2. $\log x=c_1 y/x+c_1^2 \log (y-c_1 x)-c_1^2 \log x+c_2$.
3. $y=x (c_1 \log x+c_2)^2$.
4. $c_1 \theta=\int \dfrac{ds}{[m(1+s)^2+n^2]^{1/2}-s^2-s}+c_2$

where $\dfrac{dz}{d\theta}=zs,\ y=xz,\ x=e^{\theta}$.

5. $\theta=\int \dfrac{dz}{c_1+(1-z)^2}+c_2$, where $x=e^{\theta}$, $y=zx^2$.

6. $\theta=\int (c_1 e^z-4z-2)^{-1} dz+c_2$, where $x=e^{\theta}$, $y=x^2 z$.

7. $\theta=\int (c_1 e^z+4z-1)^{-1} dz+c_2$, where $x=e^{\theta}$ $y=x^{-2} z$.

Miscellaneous Examples on Chapter VII (Page 118)

1. $\dfrac{\sqrt{2e^y+c^2}-c}{\sqrt{2e^y+c^2}+c}=c_1 e^{cx}$ or $e^{\frac{1}{2}y}=\dfrac{\sqrt{2}}{c-x}$ or $2e^y=c^2 \sec^2 (\frac{1}{2} cx+c_1)$

according as the first constant of integration is c^2, 0, or $-c^2$.

2. $y=c_1 \sin (ax+c_2)$.
3. $y=c_1 e^{x/a}+c_2 x+c_3$.
4. $y=\frac{1}{2}x^2 \log x+c_1 x^3+c_2 x^2+c_3 x+c_4$.
5. $y=(1/12) x^3+(1/16) \sin 2x+c_1 x^2+c_2 x+c_3$.
6. $3x=2a^{1/4} (y^{1/2}-2c_1) (y^{1/2}+c_1)^{1/2}+c_2$.
7. $y\sqrt{1+x^2}=c_1 \sinh^{-1} x+c_2$.
8. $(x+c_1)^2+(y+c_2)^2=a^2$.
9. $\sin [(x+c_2)\sqrt{(1+c_1)}]+\sqrt{(1/c_1+1)} \cos y=0$.
10. $y=e^{-\frac{1}{2}x^2}+c_1 x+c_2$.
11. $y=(\sin^{-1} x)^2+c_1 \sin^{-1} x+c_2$.
12. $15 c_1^3 y=4 (c_1 x+a^2)^{5/2}+c_2 x+c_3$.
13. See Q. 7. VII (E).
14. $y (x^2-4)^2=x^4-12x^2+c_1 (\frac{1}{2} x^{3/2}-4x \log x)+c_2 (x^2+4)+c_3 x$.

EXERCISE VIII (A) (Page 125)

1. $y=c_1 x+c_2 x \int x^{-2} e^{\frac{1}{3}x^3} dx+1.$

2. $y=c_1 e^x+c_2 (x^3+3x^2+6x+6)$
3. $y=e^x \log x+c_1 e^x \int x^{-1} e^{-x} dx+c_2 e^x.$
4. $y=c_1 e^x (x+1)^5+c_2 e^x-\frac{1}{4} x e^x.$
5. $y=c_1 e^x+c_2 e^{3x} (4 x^3-42 x^2+150 x-183).$

6. $y=c_1 x+c_2 x \int x^{-2} e^{-\frac{1}{2}x^2} dx$
$$+x \int x^{-2} e^{-\frac{1}{2}x^2} \int X x e^{\frac{1}{2}x^2} (dx)^2.$$

7. $y=c_1 e^x+c_2 e^{-x}+c_3 \left(e^x \int e^{\frac{1}{2}x^2-x} dx-e^{-x} \int e^{\frac{1}{2}x^2+x} dx \right)$

8. $y=c_1 x^2+c_2 x+c_3 (x^2 \int x^{-3} e^{-x} dx-x \int x^{-2} e^{-x} dx)$
9. $y=c_1 x+(x^2+c_2 x) e^x.$

10. $y=c_1 e^{ax}+c_2 e^{ax} \int e^{\frac{1}{2}ax^2-2ax} dx$

11. $y=c_1 (x^3+a)+c_2 x.$

EXERCISE VIII (B) (Pages 128-129)

1. $y=e^{-x^3} \left(c_1 e^{x\sqrt{2}}+c_2 e^{-x\sqrt{2}} \right)$

2. $y=\dfrac{c_1 \sin nx}{x}+\dfrac{c_2 \cos nx}{x}.$

3. $xy=c_1 e^{nx}+c_2 e^{-nx}.$

4. $y=c\, e^{\frac{1}{2} bx^2} \sin (x\sqrt{b}+A)$

5. $y=c\, e^{\frac{1}{2} bx^2} \sin (x\sqrt{b}+A)+e^{\frac{1}{2} bx^2} \left\{ \dfrac{1}{D^2+b} \left(x\, e^{-\frac{1}{2} x^2 b} \right) \right\}$

6. $y=c\, e^{-x^4/8} \sqrt{x} \sin (\frac{1}{2}\sqrt{3} \log x+A)$
7. $y=e^{2x} (c_1 \cos \log x+c_2 \sin \log x)$
8. $y=(c_1 \sin \sqrt{6}\, x+c_2 \cos \sqrt{6}x) \sec x+\frac{1}{7} e^x \sec x$
9. $y=\sec x\, (c_1 e^{ax}+c_2 e^{-ax})$

10. $y=c_1 \left(\dfrac{\sin nx}{nx}+\dfrac{\cos nx}{n^2 x^2} \right)+c_2 \left(\dfrac{\cos nx}{nx}-\dfrac{\sin nx}{n^2 x^2} \right)$

11. $y \sin nx=c_1 \cos mx+c_2 \sin mx$

12. $y=(c_1 x^2+c_2 x^{-1}).\ e^{\sqrt{x}}$
13. $y=x^n (c_1 \cos ax+c_2 \sin ax).$

14. $y\, e^{-x^2}=c_1 e^x+c_2 e^{-x}-1.$

EXERCISE VIII (C) (Pages 133-134)

1. $y=c_1 \sin(\sin x+c_2)$ 2. $y=c_1 \cos a/x+c_2 \sin a/x$.
3. $y=c_1 \sin(n\sqrt{x^2-1}+\alpha)$.
4. $y=c_1 \cos(m \sin^{-1} x)+c_2 \sin(m \sin^{-1} x)$
5. $y=c_1 e^{-\cos x}+c_2 e^{\cos x}$
6. $y=c \cos(\log \tan \frac{1}{2} x+A)$
7. $y=c_1 \cos(2 \tan^{-1} x)+c_2 \sin(2 \tan^{-1} x)$
   or $y(1+x^2)=c_1(1-x^2)+c_2 x$.
8. $y=c_1 e^{(n-1)\tan x}+c_2 e^{-n \tan x}$.
9. $y=c_1 e^{2\cos x}+c_2 e^{\cos x}+e^{-\cos x}/6$.

EXERCISE VIII (D) (Pages 135-136)

1. $y=c_1 e^{2x}+c_2 e^{2x}\int e^{2/3x-2x}dx$ 2. $y=x^3+c_1x^2+c_2 x+c_3+c_4 e^{-x}$.
3. $y=c_1 e^{x}+c_2 e^{x}\int e^{-2x+x^{-1}}dx-\frac{1}{2}e^{-x}$
5. $y=-e^{x}-\frac{1}{4}c(2x+5)+b e^{2x}$

EXERCISE VIII (E) (Page 140)

1. $y=a \cos x+b \sin x+x$,
2. $y=(a-x)\cos x+(b+\log \sin x)\sin x$.
3. $y=[a-\log \tan(\frac{1}{4}\pi+x)]\cos 2x+b \sin 2x$.
4. $y=a e^{x}+bx+(x^2+x+1)$.
5. $y=[a-e^{-x}+\log(1+e^{-x})]e^{x}+[b-\log(1+e^{x})]e^{-x}$
6. $y(1-x^2)=a \cos x+b \sin x+x$.
7. $y=x^3+x+\frac{1}{2}-\frac{2}{3}x^4+c_1 xe^{2x}+c_2x$ 8. $y=a e^{x}+bx-(1+x+x^2)$

Miscellaneous Examples on Chapter VIII (Pages 141-144)

2. (i) $2y=x(c_1 e^{2x}+c_2-x)$ (ii) $y=c_1(x^2-a)+c_2 x$
   (iii) $y=cx \sin(nx+\alpha)$ (iv) $y=x+(c_1 x+c_2)e^{-x^2/2}$
   (v) $y=c_1 \sin\left(\sqrt{\dfrac{a^2-x^2}{a}}+\alpha\right)$
   (vi) $y=c \cos\left(\dfrac{n+\alpha x}{x}\right)$
   (vii) $y=\cos\left\{\dfrac{a}{2}\log\dfrac{1+x}{1-x}+\alpha\right\}$
   (viii) $y e^{-x}=-x+\int(2x-1)\log(2x-1)\,dx$
   $-\int(2x-1)e^{-2x}\int 2e^{2x}\log(2x-1)(dx)^2-c_1 x e^{-2x}+c_2$.
   (ix) $y=c_1 x+c_2 x^{-1}+x^3$.
   (x) $y=e^{-1/(2x)^2}[c \cos(2x+\alpha)+\frac{1}{4}x]$
3. $y=(1-x^2)(c_1+c_2 \log x)$; $y=(1-x^2)(x+c_1+c_2 \log x)$.

4. $y=x^2+A\left\{\sin(x+a)+\frac{1}{x}\cos(x+a)\right\}$

5. (i) $y=(c_1 x^2+c_2 x^{-3})/(x-2)$.

(ii) $y=e^{2x}+(c_1 x^3+c_2)\, e^x$ (iii) $y=c/x+c_1(x+1/x)$

(iv) $y=c_1 x^3+c_2 x^{-3}$

(v) $y=c_1 e^{a\sin^{-1}x}+c_2 e^{-a\sin^{-1}x}$

(vi) $y=c_1 x^2+c_2 \sin x$

(vii) $y=c_1 \sin(x^2+a)+\frac{1}{4}x^2$ (viii) $y=e^{2x}$

(ix) $y=c_1 \cos(m\sin^{-1}x)+c_3 \sin(m\sin^{-1}x)$

(x) $y=c_1 \cos[2(1+x)\, e^{-x}]+c_2 \sin[2(1+x)\, e^{-x}]+(1+x)e^{-x}$

(xi) $y=1+e^{-x^2/2}$

6. (i) $y=A'(2x+5)+B' e^{2x}-e^x$

(ii) $y=c_1 x+c_2(x\sin^{-1}x+\sqrt{1-x^2})-\frac{1}{9}x(1-x^2)^{3/2}$

9. $I=-\frac{1}{4}$.

EXERCISE IX (A) (Pages 153-154)

1. $x=c_1\cos\omega t+c_2\sin\omega t$: $y=c_1\sin\omega t-c_2\cos\omega t$;
locus is $x^2+y^2-c_1^2+c_2^2$, a circle.

2. $x=c_1\cos t+c_2\sin t$; $y=-\frac{1}{2}(c_1+3c_2)\sin t+\frac{1}{2}(c_2-3c_1)\cos t$.

3. $x=c_1 e^{-2t}+c_2 e^{-7t}-31/196+5/14\, t-\frac{1}{8}e^t$;
$y=-\frac{2}{3}c_1 e^{-2t}+c_2 e^{-7t}+9/98-1/7\, t+5/24\, e^t$.

4. $x=c_1 e^{-4t}+c_2 e^{-7t}+7/40\, e^t+1/27\, e^{2t}$,
$y=\frac{1}{2}c_1 e^{-4t}-c_2 e^{-7t}+1/40\, e^t+7/54\, e^{2t}$.

5. $x=c_1 e^{-t}+c_2 e^{-6t}+19/3\, t-56/9-29/7\, e^t$.
$y=-c_1 e^{-t}+4c_2 e^{-6t}-17/3\, t+55/9+24/7\, e^t$.

6. $x=c_1 e^{-11/8t}+3t-2$, $y=3+5t+6/19\, c_1 e^{-11/8t}$.

7. $x=1/10\, e^{-t}(10\cos 2t-\sin 2t)-4/5\, e^{-3t}$.
$y=1/10\, e^{-t}(21\sin 2t-8\cos 2t)+4/5\, e^{-3t}$.

8. $x=e^t(A+Bt)+e^{-t}(C+Dt)-23$,
$y=e^t(A_1+B_1 t)+e^{-t}(D_1+D_1 t)+18$
where $A_1=\frac{1}{2}(B-A)$, $B_1=-\frac{1}{2}B$, $C_1=-\frac{1}{2}(C+D)$, $D_1=-\frac{1}{2}D$

9. $x=(c_1+c_2 t)\, e^t+c_3 e^{-3/2t}-\frac{1}{2}t$,
$y=-2\,(\,(c_1+c_2 t-3c_2)\, e^t-\frac{1}{3}c_3 e^{-3/2t}-\frac{1}{8}$.

10. $x=e^{-4t}(c_1\cos t+c_2\sin t)+31/26\, e^t-93/17$,
$y=-(c_1+c_2)\, e^{-4t}\cos t+(c_1-c_2)\, e^{-4t}\sin t-2/13\, e^t+6/17$.

11. $x=(c_1+c_2 t)\, e^{2t}+c_3 e^{-2t}-t$,
$y=(3c_2-2c_1-2c_2 t)\, e^{2t}-\frac{1}{3}c_3 e^{-3t}-\frac{1}{8}$.

12. $y=(c_1+c_2 x)\, e^x+3c_3 e^{-3/2x}-\frac{1}{2}x$
$z=2(3c_2-c_1-c_2 x)\, e^x-c_3 e^{-3/2x}-\frac{1}{8}$.

13. $x=c_1 t+c_2 t^{-1}$, $y=-c_1 t+c_2 t^{-1}$.

14. $x=c_1\cos\sqrt{3}t+c_2\sin\sqrt{3}t+c_3\cos\sqrt{2}t$
$+c_4\sin\sqrt{2}t-49/1452\, e^{3t}+5/66t\, e^{3t}-1/12-\frac{1}{4}\cos 2t$
$y=-3c_1\cos\sqrt{3}t-c_2\sin\sqrt{3}t-2c_3\cos\sqrt{2}t$
$-2c_4\sin\sqrt{2}t+1/66t\, e^{3t}+\frac{1}{3}-23/1452\, e^{3t}$.

15. $xt=c_1 \cos t+c_2 \sin t,\ yt^2=c_3+2\,(c_2 \cos t-c_1 \sin t)$
$+t\,(c_1 \cos t+c_2 \sin t)$.

16. $(l_1\,x+m_1\,y+n_1\,z+r_1\,\lambda_1^{-1})^{\lambda_1^{-1}}$

$$=A\,(l_2\,x+m_2\,y+n_2\,z+r_2\,\lambda_2^{-1})^{\lambda_2^{-1}}$$

$$=B\,(l_3\,x+m_3\,y+n_3\,z+r_3\,\lambda_3^{-1})^{\lambda_3^{-1}},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of $\begin{vmatrix} a-\lambda & a' & a'' \\ b & b'-\lambda & b'' \\ c & c' & c''-\lambda \end{vmatrix}=0$,

$r=ld+md'+nd''$ and l, m, n be so chosen that
$al+a'm+a''n=\lambda_1\,l,\ bl+b'm+b''n=\lambda_2 m,\ cl+c'm+c''n=\lambda_3\,n.$

EXERCISE IX (B) (Pages 160-161)

1. $x=az,\ y=bz$ 2. $x^2=z^2+a,\ x^3=y^3+b$
3. $x-y=a\,(z-x),\ (x-y)^2\,(x+y+z)=b.$
4. $x^3-y^3=a^3,\ x^3+3/z=b$ 5. $y=az,\ x^2+y^2+z^2=bz.$
6. $z=a\,(2+x+y),\ z\,(x-y)=b.$
7. $y-x=axy,\ z=\dfrac{nxy}{y-x}\log\dfrac{y}{x}+b$
8. $y^2+z^2=a,\ \log bx=\tan^{-1}\dfrac{y}{z}$
9. $l^2\,x+m^2\,y+n^2\,z=a,\ l^2\,x^2+m^2\,y^2+n^2\,z^2=b.$
10. $z-x^3\sin\,(y+2x)=b,\ y+2x=a.$
11. $x^2+y^2+z^2=a,\ y^2-2\,yz-z^2=b.$
12. $y-3x=a,\ 5z+\tan\,(y-3x)=b\,e^{5x}$
13. $y=Ax,\ x^{1-a}=B\,[\,z+\sqrt{x^2+y^2+z^2}]$
14. $yz=ax,\ x^2+y^2+z^2=b$ 15. $xy=a,\ (z^2+xy)^2-x^4=b.$
16. $xy=a,\ x^2+y^2+(x+y)\,z=b.$
17. $y-x=c_1\,(z-x),\ (y-x)=c_2\,(z-y).$
18. $x^2\,(x^2-y^2)=-2/z+2b.$

EXERCISE X (A) (Pages 167-168)

1. $x+y+z=c$ 2. $x^2+y^2+z^2=c^2$ 3. $xyz=c$
4. $e^{x}\,(y+z)=c$ 5. $x-cy-y\log z=0$ or $z\,e^{-x/y}=c$
6. $x^2+2\,yz+2\,z^2=c^2.$ 7. $xy+yz+zx=c^2$
8. $xy^2=cz^3$ 9. $xy=c\,(a+z)$
10. $a/x+b/y+c/z=c_1$ 11. $(x+y+z)\,e^z=c$
12. $xyz+x^2+y^2+z^2=c$ 13. $x^2+y^2+z^2=cx$
14. $x^2y=cz\,e^z$ 15. $(x+a)\,(y+b)\,(z+c)=A$
16. $\sqrt{x^2+y^2+z^2}+\tan^{-1}x/z+ax^3+by^2+cz=A$

17. $z=(x+c)(y+a)$
18. $ax-cz=A\,(ay-bz)$
19. $ny-mz=c\,(nx-lz)$
20. $(x+y)^2\,(y+z)=c.$
21. $(x+2y)\,(y-z)^2=c$
22. $x^2+2y^2-6xy-2xz+z^2=b$
23. $x^2\,y^2+a\,x^4\,z^2+y^3+z^2+\sqrt{y^2+z^2}=c$
24. $x^2\,y+y^2\,z+\log\,(x+y+z)=c$
25. $f(y)=ky,\ x^k=c\,y^z$
26. $y=\sin x+cz\,e^{-x}.$
27. $\sqrt{x^2+y^2+z^2}+\tan^{-1} x/z=c.$

EXERCISE X (B) (Page 172)

1. $y^2-yz-xz=cz^2$
2. $x^2-xy+y^2=cz$
3. $x^2\,y+y^2\,z+z^2\,x=c$
4. $xyz\,(x+y-z)=c.$
5. $z/x+x/y+y/z=c.$
6. $x^2+y^2+z^2=c\,(x+y+z)$
7. $x^2\,(y+z)\,(2y-z)=c.$

EXERCISE X (C) (Page 176)

1. $(y+z)\,e^{x+y}=ct$
2. $x^2+xy^2+x^2\,t-xz=c$
3. $(x-c_1)\,(y-c_2)\,(z-c_3)=0$
4. $(x^2+y^2+z^2-c_1)\,[\,(x^2+y^2)\,z^2-c_2\,]=0$
5. $(lx+my+nz-c_1)\,(l'\,x+m'\,y+n'\,z-c_2)=0$

EXERCISE XI (A) (Page 181)

1. $\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}=1.$
2. $q=2y\,p^2.$
3. $z=pq.$
4. $z^2\,(p^2+q^2+1)=a^2$
5. $q=px+p^2.$

EXERCISE XI (B) (Pages 182-183)

1. $b\,\dfrac{\partial z}{\partial x}+a\,\dfrac{\partial z}{\partial y}=2abz$
2. $y\,p-x\,q=0$
3. $p\,x+q\,y=0$
4. $x^2\,\dfrac{\partial z}{\partial x}+y\,\dfrac{\partial z}{\partial y}=2y^2$
5. $\dfrac{\partial^2 z}{\partial y^2}=a^2\,\dfrac{\partial^2 z}{\partial x^2}$

EXERCISE XI (D) (Page 192)

1. $\phi\,(x^2-y^2,\ x^2-z^2)=0$
2. $\phi\,[\,(x-y)^2\,(x+y+z),\ (x-y)/(y-z)\,]=0$
3. $\phi\,[\,y+x,\ \log\,(z^2+y^2+2y\,x+x^2)-2x\,]=0$
4. $z^2-xy=\phi\,\left(\dfrac{y}{x}\right)$

5. $z=\frac{nxy}{y-x}\log\frac{y}{x}+\phi\left(\frac{y-x}{xy}\right)$

6. $(x+y)\log z=x\cdot\phi(x+y)$

7. $z=\sqrt{x^2+y^2}\cdot\phi\left[\tan^{-1}\frac{y}{x}+\frac{1}{2}\log(x^2+y^2)\right]$

8. $\frac{\sin z}{\sin y}=\phi\left(\frac{\sin x}{\sin y}\right).$ 9. $\log z=-\frac{ax}{3y^2}+\phi(xy)$

10. $(x-y+z)^2=(x+y-z)\,\phi(x-3y-z)$

11. $xyz-3u=\phi\left(\frac{y}{x},\frac{x}{z}\right)$

12. $z^2-x_1^2=\phi(x_2^2-x_1^2,\ x_3^2-x_1^2).$

EXERCISE XI (E) (Page 196)

1. $z=ax+\frac{y}{a}+c$ 2. $z=x\sec\alpha+y\tan\alpha+b.$

3. $z=x(1+a)+y\left(1+\frac{1}{a}\right)+c.$

4. $Z=aX+(1-a^m)^{1/n}Y+b$
where $Z=z^{(m-n-1)/(m-n)}$, $Y=\frac{1}{2}(y-\frac{1}{2}\sin 2y)$,
$X=\frac{1}{2}(x+\frac{1}{2}\sin 2x)$.

EXERCISE XI (F) (Page 198)

1. $4c(z-a)=(x+cy+b)^2+4.$ 2. $z^2(1+a^2)=8(x+ay+b)^2.$
3. $(a^2+1)(c^2-z^2)=(x+ay+b)^2$
4. $z^{2m-1}=(c_1x+c_2y+c_3)^2$, where $4m^2c_1^m c_2^m=(2m-1)^2$

5. $(z+a^2)^3=(x+ay+c)^2$ 6. $z^{\sqrt{1+a}}=cxy^a$

EXERCISE XI (G) (Page 199)

1. $2az=a^2x^2+y^2+2ab$
2. $z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y-a)^{3/2}+b$
3. $az=ax^2+a^2x+e^{ay}+ab.$ 4. $(2z-ay^2-2b)^2=16ax$

EXERCISE XI (H) (Page 201)

1. $z=ax+by+\log ab$ 2. $z=ax+by+a^2+b^2$
3. $z=ax+by-2\sqrt{ab}.$ 4. $z=ax+by-n\,a^{1/n}\,b^{1/n}.$

EXERCISE XI (I) (Page 207)

1. $z=ax\,e^y+\frac{1}{2}a^2e^{2y}+b.$ 2. $z=ax+a^2y^2+b$
3. $z^{3/2}=(x+a)^{3/2}+(y+a)^{3/2}+b.$

4. $2\sqrt{z}=cx+y/c+b$

5. $2z=x^2+y^2+x\sqrt{x^2+a^2}+y\sqrt{y^2-1-a}$

$$+\log\frac{[\,x+\sqrt{x^2+a^2}\,]^a}{[\,y+\sqrt{y^2-1-a}\,]^{1+a}}+b.$$

6. $\sqrt{1+a}\; z=2\sqrt{x+ay}+b.$

7. $\frac{1}{2}z^2=-\dfrac{1}{a^2 x}-\dfrac{2a}{\sqrt{y}}+b.$ 8. $\log\dfrac{x^a}{bz}=\dfrac{1}{2a^2 y^2}.$

9. $z=\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+\dfrac{x\sqrt{a^2-x^2}}{2}-\dfrac{a^2}{y}-y+b.$

10. $z=a\,y+b\,(x^2-a).$

Miscellaneous Exercise on Chapter XI (Pages 207-208)

1. (*i*) $\{\cos(x+y)+\sin(x+y)\}\,e^{y-x}$

$$=\phi\left[z^{\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right)\right]$$

(*ii*) $f\{(x^2-z^2),\,(x^3-y^3)\}=0$

(*iii*) $lx+my+nz=\phi\,(x^2+y^2+z^2)$

(*iv*) $x+y+z=\phi\,(xyz)$

(*v*) $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)=f\left(\dfrac{1}{x}-\dfrac{1}{z}\right)$

(*vi*) $(x+y+z)=\phi\,(xyz)$

(*vii*) $\dfrac{1}{(x-y)^2}-\dfrac{1}{(x+y)^2}=F\left(\dfrac{z^2}{xy}\right)$

(*viii*) $\phi\left(\dfrac{x-y}{y-z},\,\dfrac{y-z}{z-x}\right)$

(*ix*) $xyz=\phi\,(yz+zx+xy)$

(*x*) $\phi\left(\dfrac{x-y}{y-z},\,\dfrac{z-x}{y-z}\right)=0$

(*xi*) $\phi\left(\dfrac{y}{z},\,\dfrac{x^2+y^2+z^2}{y+z}\right)=0$

(*xii*) $x^2+y^2+z^2=z\phi\left(\dfrac{y}{z}\right)$

(*xiii*) $z=\dfrac{1}{x^3 y^3}\,\phi\left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)$

(*xiv*) $\phi\,(xy-z^2,\ x^2-y-z)$.

3. (*i*) $2z=\left(\dfrac{x}{a}+ay\right)^2+b.$

(*ii*) $z=xy+y\sqrt{x^2-a_1^2}+b_1,\ z=xy+x\sqrt{y^2+a_2^2}+b_2.$

4. (*i*) $z=mx+y\,e^{-m/a}+c$

(*ii*) $az-1=c\,e^{x+ay}$

(*iii*) $xz=ay+2\sqrt{ax}+b$

(*iv*) $z=a\left(x+\dfrac{3}{2+\sqrt{10}}y\right)+c$

(*v*) $z^2=x\sqrt{x^2+a^2}+y\sqrt{y^2-a^2}+a^2\log\dfrac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}+b$

(*vi*) $z=ax+b\,e^y\,(y+a)^{-a}+c$

(*vii*) $z^2\pm\left[z\sqrt{z^2-4a^2}-4a^2\log\left(z+\sqrt{(z^2-4a^2)}\right)\right]=4(x+ay+b).$

(*viii*) $z=ax+by+\sqrt{\alpha a^2+\beta b^2+\gamma}$

(*ix*) $(a^2+z^2)^{3/2}=3\,(x+ay)+c$

(*x*) $\dfrac{2}{2-l}\,z^{1-l/2}=\dfrac{ax^{m+1}}{m+1}+\dfrac{y^{n+1}}{a(n+1)}+b.$

(*xi*) $z-B=\frac{1}{2}\int\left\{\frac{3}{2}\,(x+y)^2+A\right\}^{1/2}(dx+dy)$
$+\frac{1}{2}\int\left\{\frac{1}{2}\,(x-y)^2-A\right\}^{1/2}(dx-dy).$

(*xii*) $z=a\sqrt{x+y}+\sqrt{1-a^2}\sqrt{x-y}+b.$

(*xiii*) $(2z-b)=a\,(x+y)-x^2-y^2+\frac{1}{2}\,(x-y)\left\{a^2+2\,(x-y)^2\right\}^{1/2}.$
$+\dfrac{a^2}{2\sqrt{2}}\log\left[2\,(x-y)+\sqrt{2}\left\{a^2+2\,(x-y)^2\right\}^{1/2}\right]$

(*xiv*) $z=A+x\cos\alpha+\frac{1}{2}\cos\alpha\log\cosh 2y+\sin\alpha\tan^{-1}e^{2y}.$

5. (*viii*) $z=b.$ (*ix*) $\dfrac{x^2}{\alpha}+\dfrac{y^2}{\beta}+\dfrac{z^2}{\gamma}=1.$